Mathematical Contributions of Archimedes: Some Nuggets

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Archimedes is generally regarded as the greatest mathematician of antiquity and alongside Isaac Newton and C F Gauss as the top three of all times. He was also an excellent theoretician-cum-engineer who identified mathematical problems in his work on mechanics, got hints on their solution through engineering techniques and then solved those mathematical problems, many a time discovering fundamental results in mathematics, for instance, the concepts of *limits* and *integration*. In his own words, " *which I first discovered by means of mechanics and then exhibited by means of geometry*". In this article we briefly describe some of his main contributions to mathematics.

1. Works of Archimedes

In all, Archimedes seems to have composed ten treatises. Some of these are available in fragments, some only in translations and some as parts of commentaries. The story of how his various works were discovered is quite fascinating. Here is a list of the ten treatises (the titles are given in English translations):

1. *On the Equilibrium of Planes* (in two Books);
2. *On the Floating Bodies* (in two Books);
3. *Sand Reckoner*;
4. *On the Measurement of the Circle*;
5. *On Spirals*;
6. *Quadrature of the Parabola*;
7. *On Conoids and Spheroids*;
8. *On the Sphere and Cylinder* (in two Books);

Keywords
Sphere, parabola, Archimedean solids, polyhedron.

9. *Book of Lemmas*;
10. *The Method.*

Considering that there are several results attributed to Archimedes (by many authors later on) but which are not found in the above treatises, we can only conclude that he must have composed other treatises, traces of which have completely vanished. Principal among them is the discovery of *Archimedean Solids*; the formula for the area of a triangle, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, which is generally attributed to Heron of Alexandria (who lived several centuries after Archimedes) is credited to Archimedes by Arabic scholars; similarly, the theorem on the broken chord (see Exercise 1, on p.16).

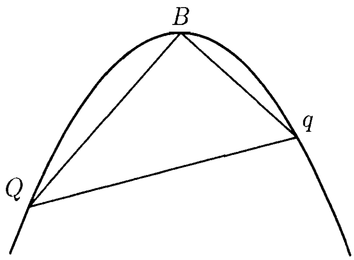
The first two treatises mentioned above fall into the physics books category. Unlike the 8-Book treatise of Aristotle, *Physics*, which is about a century older and which is speculative and nonmathematical, Archimedes' treatment is along the lines of Euclid's *The Elements.* He starts off from a set of simple postulates and develops deep results. (It is interesting to recall that one of the *Hilbert's twenty problems* is the axiomatisation of Physics which is what Archimedes seems to have attempted.)

2. Some Well-Known Results of Archimedes

Here we shall list a few of the well-known results found in his various treatises. In later sections we discuss a couple of them in detail.

i. *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*

ii. *The ratio of the circumference of any circle to its diameter is less than* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$.



iii. *Any segment of a sphere has to the cone with the same base and height the ratio which the sum of the radius of the sphere and the height of the complementary segment has to the height of the complementary segment.*

iv. *Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and same height.*

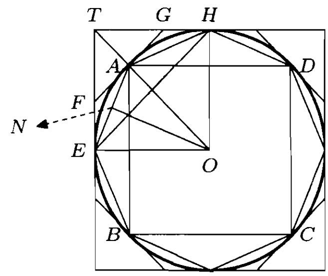
The proofs of results (iii) and (iv) are especially marvellous which he discovered by using law of the lever and similar balancing properties. Archimedes deduced from result (iii) the volume of a sphere since volumes of cones and cylinders were known by then. (Though tempting, I am not including the proofs of these two results as the article is already long; perhaps one could take it up in a separate article later.)

In this article I will describe the proofs of the first two propostions which are contained in *On the Measurement of the Circle.* Especially, the second is more interesting as it gives the most popular value of π used, $\pi \approx 22/7$. Though the methods are due to Archimedes, I shall describe the proofs in a language and notation familiar to us.

3. Area of a Circle

In *Proposition I* of *On the Measurement of the Circle* Archimedes gives the formula for the area of a circle as half of the product of its circumference and radius (which takes the form familiar to us, πr^2, since π is the ratio of the circumference to the diameter of a circle). The method he uses to prove this is quite ingenious – he proves that it can neither be more nor be less than the claimed quantity.

Let r be the radius of a circle and C its circumference.



The claim is that

$$\text{area of the circle} = \frac{rC}{2}.$$

Suppose the area of the circle is greater than $rC/2$. In the (slightly modified) words of Archimedes himself (T L Heath's translation [1]):

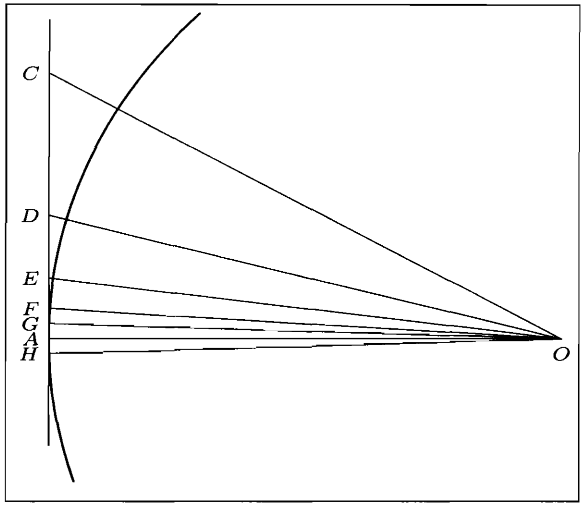
Inscribe a square $ABCD$, bisect arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments sum of whose areas is more than $rC/2$ (which can be done since the area of the circle is assumed to be more than $rC/2$). Thus the area of the polygon is greater than $rC/2$. Let AE be any side of it and ON the perpendicular on AE from O. Then ON is less than the radius of the circle and the perimeter of the polygon is less than the circumference of the circle. Therefore the area of the polygon is less than $rC/2$ which is consistent with the hypothesis.

Thus the area of the circle is not greater than $rC/2$. Similarly, he proves that the area cannot be less than $rC/2$ by considering polygons circumscribing the circle and thus concludes that the area must be equal to $rC/2$.

4. Approximations of π

Archimedes proves that the value of π lies between $3\frac{10}{71}$ and $3\frac{1}{7}$. The proof is so simple that it is a sin not to introduce it in the school curriculum. This proof will go a long way in clearing the mystery built around π. The only tools he uses are the *Bisector theorem*[1] and the *Pythagoras* theorem and of course two approximations to $\sqrt{3}$, $\sqrt{3} > 265/153$ and $\sqrt{3} < 1351/780$, which he seems to produce by magic. You need to keep paper and pencil ready (and perhaps a calculator too) to verify the calculations involved in the proof.

[1] An angle bisector in a triangle divides the side opposite to that angle in the ratio of the sides containing the angle.



Proof of $\pi < 3\frac{1}{7}$: Let AO be the radius of a circle, O its centre, AC the tangent to the circle at A such that $\angle AOC = 30°$ Then we have

$$\frac{OA}{AC} = \frac{\sqrt{3}}{1} > \frac{265}{153}, \quad \frac{OC}{AC} = \frac{2}{1} = \frac{306}{153}. \tag{1}$$

Note the use of the first magic approximation to $\sqrt{3}$ of Archimedes.

Let OD be the bisector of $\angle AOC$. By the *Bisector theorem*

$$\frac{OC}{OA} = \frac{CD}{AD}, \quad \text{and so} \quad \frac{OC + OA}{OA} = \frac{AC}{AD} \quad \text{or}$$

$$\frac{OC + OA}{AC} = \frac{OA}{AD}.$$

Using (1) we get

$$\frac{OA}{AD} > \frac{571}{153}.$$

Now he relates OD and AD using *Pythagoras theorem*:

$$\frac{OD^2}{AD^2} = \frac{OA^2 + AD^2}{AD^2} > \frac{571^2 + 153^2}{153^2} = \frac{349450}{23409}$$

which on taking square roots gives

$$\frac{OD}{AD} > \frac{591\frac{1}{8}}{153}.$$

Let OE bisect $\angle AOD$ and OF bisect $\angle AOE$. Carrying on as before we obtain

$$\frac{OE}{AE} > \frac{1172\frac{1}{8}}{153}, \quad \frac{OF}{AF} > \frac{2339\frac{1}{4}}{153}.$$

Consider the bisector OG of $\angle AOF$ We have

$$\frac{OA}{AG} > \frac{2334\frac{1}{4} + 2339\frac{1}{4}}{153} = \frac{4673\frac{1}{2}}{153}.$$

Thus the $\angle AOC$ has been bisected successively four times and therefore

$$\angle AOG = \frac{90^\circ}{3 \times 16}.$$

Let H be the point on CA beyond A such that $AH = AG$. Then $\angle GOH = 90^\circ/24$ which means that GH is a side of a polygon of 96 sides circumscribed about the given circle. Since diameter $= 2OA$ and $GH = 2AG$ it follows that

$$\frac{\text{diameter}}{\text{(perimeter of the 96-gon)}} > \frac{4673\frac{1}{2}}{153 \times 96} = \frac{4673\frac{1}{2}}{14688}.$$

Thus the circumference of the circle being less than the perimeter of the 96-gon we get

$$\frac{\text{circumference of the circle}}{\text{diameter of the circle}} < \frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} <$$

$$3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} = 3\frac{1}{7}.$$

Thus we obtain $\pi < 22/7$.

Proof of $\pi > 3\frac{10}{71}$: Let AB be the diameter of a circle, and C be a point on the circle such that $\angle BAC = 30°$. Consider the triangle ABC. We have

$$\frac{AC}{CB} = \frac{\sqrt{3}}{1} < \frac{1351}{780}. \qquad (2)$$

(Note the apppearence of the second magical approximation to $\sqrt{3}$.) Let D be a point on the circle such that AD bisects $\angle BAC$ and X be the point of intersection of AD and BC.

Then the triangles ADB, ACX, and BDX are similar. Therefore we have

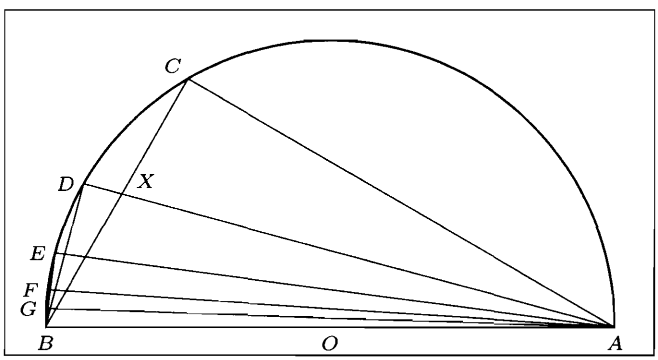
$$\frac{AD}{DB} = \frac{BD}{DX} = \frac{AB}{BX} = \frac{AC}{CX} = \frac{AB+AC}{BX+CX} = \frac{AB+AC}{BC}.$$

(Note the use of componendo-dividendo.) Now using the approximation to $\sqrt{3}$ given above and noting that $AC = 2BC$ we get

$$\frac{AD}{DB} < \frac{1351}{780} + \frac{1}{2} = \frac{2911}{780}.$$

We now need to relate AB and BD. Just as in the previous case:

$$\frac{AB^2}{DB^2} = \frac{AD^2 + DB^2}{DB^2} = \frac{AD^2}{DB^2} + 1 < \frac{2911^2}{780^2} + 1 = \frac{9082321}{608400}$$



and so

$$\frac{AB}{DB} < \frac{3013\frac{3}{4}}{780}.$$

Proceeding this way, constructing bisectors of $\angle BAD$, $\angle BAE$, $\angle BAF$ to get the points E, F G on the circle and repeating the above calculations successively, we get

$$\frac{AB}{EB} < \frac{1838\frac{9}{11}}{240}, \quad \frac{AB}{FB} < \frac{1009\frac{1}{6}}{66} \quad \text{and} \quad \frac{AG}{GB} < \frac{2016\frac{1}{6}}{66}.$$

Note that here BG is a side of a regular 96-gon inscribed in the circle. We need an estimate of the ratio BG/AB:

$$\frac{AB^2}{BG^2} = \frac{AG^2 + BG^2}{BG^2} < \frac{(2016\frac{1}{6})^2 + 66^2}{66^2}$$

giving us

$$\frac{BG}{AB} > \frac{66}{2017\frac{1}{4}}.$$

Therefore, finally

$$\frac{\text{perimeter of 96-gon}}{AB} > \frac{66 \times 96}{2017\frac{1}{4}} > 3\frac{10}{71}.$$

5. Archimedean Solids

While the *Platonic Solids* or the *regular polyhedra* – tetrahedron, cube, octahedron, dodecahedron and icosahedron – are well-known, the *Archimedean Solids* or the *semi-regular polyhedra* do not seem to be well known. The most popular of the Archimedean solids is the *football*! (See [2]). Recall that a regular polyhedron is one in which the faces are congruent polygons and at each vertex the number of faces meeting is the same. A semi-regular polyhedron is one in which the faces are regular polygons not all of which are congruent and the geometrical pattern of angles at each vertex is the same. Prisms and anti-prisms are also examples of polyhedra and if we exculde this infinite family there are only 13 such solids. These are known as Archimedean solids.(See cover page.)

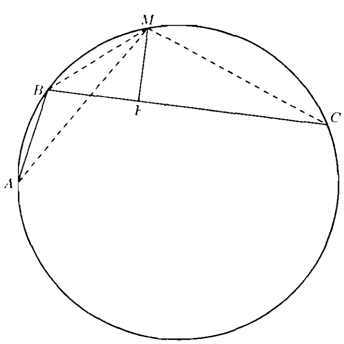
A semi-regular polyhedron is one in which the faces are regular polygons not all of which are congruent and the geometrical pattern of angles at each vertex is the same.

	Name of the Solid	V	E	F
1)	Truncated tetrahedron	12	18	$F_3 = 4$, $F_6 = 4$
2)	Cuboctahedron	12	24	$F_3 = 8$, $F_4 = 6$
3)	Truncated cube	24	36	$F_3 = 8$, $F_8 = 6$
4)	Truncated Octahedron	24	36	$F_4 = 6$, $F_6 = 8$
5)	Small Rhombicuboctahedron	24	48	$F_3 = 8$, $F_4 = 18$
6)	Great Rhombicuboctahedron or Truncated Cuboctahedron	48	72	$F_4 = 12$, $F_6 = 8$, $F_8 = 6$
7)	Snub cube	24	60	$F_3 = 32$, $F_4 = 6$
8)	Icosidodecahedron	30	60	$F_3 = 20$, $F_5 = 12$
9)	Truncated Dodecahedron	60	90	$F_3 = 20$, $F_{10} = 12$
10)	Truncated icosahedron or *football*	60	90	$F_5 = 12$, $F_6 = 20$
11)	Small Rhombicosidodecahedron	60	120	$F_3 = 20$, $F_4 = 30$, $F_5 = 12$
12)	Great Rhombicosidodecahedron or Truncated icosidodecahedron	120	180	$F_4 = 30$, $F_6 = 20$, $F_{10} = 12$
13)	Snub Dodecahedron	60	150	$F_3 = 80$, $F_5 = 12$

Table 1. Archimedean Solids.

Constructing such polyhedra out of cardboard could develop into a very addictive hobby. The book *Polyhedron Models* by Magnus J Wenninger is a good reference [3]; also, see the site http://www.korthalsaltes.com/.

Table 1 gives a list of the 13 Archimedean solids; in the table, **V** denotes the number of vertices; **E**, the number of edges; **F** the number of faces and F_m denotes a regular polygon of m sides.



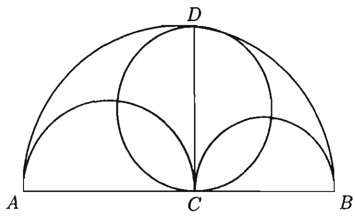
6. Exercises

Here are two Problems/Lemmas from the *Book of Lemmas*. Readers are encouraged to provide solutions/proofs.

1. *Theorem of the Broken Chord:* Suppose AB and BC are two chords of a circle with $AB \neq BC$. Let M be the midpoint of the arc ABC of the circle

and F the foot of the perpendicular from M onto the longer of the two chords. Then F divides the *broken chord* ABC equally.

2. *Arabelos* or *Shoe-maker's Knife*: Consider three points A, C, B on a line and erect semicircles with AB, AC and CB as diameters. The region that lies inside the bigger semicircle but outside the two smaller semicricles is referred to as the *Arabelos*. Show that the area of the Arabelos is equal to the area of the circle with CD as diameter where CD is the perpedicular at C. Consider the circles one of which is tangent to CD and the semicircles on AC and AB and the other which is tangent to CD and the semicircles on BC and AB. Show that these two circles are congruent.



3. Continue the iteration of Archimedes to obtain-better inequalities for π. It would be interesting to write a code to implement this procedure.

Suggested Reading

[1] T L Heath, *The Works of Archimedes*, Dover, 1953.

[2] A R Rao, The Football, *Resonance*, Vol.6, No.1, 2001.

[3] Magnus J Wenninger, *Polyhedron Models*, Cambridge University Press, 1974.

[4] Carl Boyer and Uta C Merzbach, *A History of Mathematics*, John Wiley and Sons, 1989.

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If I have ever made any valuable discoveries, it has been owing more to patient attention, than to any other talent.

Isaac Newton
(1642 – 1727)