Basis Properties of Third Order Magic Squares

Shailesh A Shirali
Amber Valley Residential School
Chikmagalur
Karnataka 577 133, India.
Email:
shailesh_shirali@rediffmail.com

Preliminaries

A 3×3 matrix is said to have the *full-magic property* if the sums of the elements in its three rows, three columns and two diagonals are all the same. The common sum is called its *magic sum*, and the matrix itself is called a *full-magic square*. (This is commonly referred to as just the *magic property*, but we wish to distinguish it from the *semi-magic property*, in which only the row and column sums are required to be the same; the diagonal sums may differ.) Traditionally, two further restrictions are placed on a full-magic square; namely, that the entries must be non-negative integers, and must be distinct. However, we shall not impose these restrictions here, unless otherwise stated. We denote the matrix by M and its magic sum by S_M, and write it thus:

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{1}$$

Let $x = b_2$ be the number in the central square. In Iyer's article (see [2]) the following are shown: (**I**) $S_M = 3x$; (**II**) for a given set of nine distinct non-negative integers in arithmetic progression, there is just one way of arranging them to make a full-magic square, not counting rotations and reflections. If rotations and reflections are taken into account, then there are precisely eight ways.

Reduced Full-Magic Squares

We now define, after Xin (see [3]), the term *reduced full-magic square*. If we subtract the least element of the square from every element, the full-magic property is preserved; likewise if we rotate the square about the center, or reflect it in any of its lines of symmetry. By

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A 'reduced full-magic square' is one in which if we subtract the least element of the square from every element, the full-magic property is preserved; likewise if we rotate the square about the center, or reflect it in any of its lines of symmetry.

performing these two actions appropriately, we can ensure that the following properties hold for M:

(A) One of the entries of M is 0.

(B) $c_3 < a_1, a_3, c_1$ and $c_1 < a_3$. This is done by rotating the square about the center to bring the least corner entry to the $(3,3)$ position, and then (if necessary) reflecting in the main diagonal.

Since $a_1 + c_3 = a_3 + c_1$, condition (B) can be refined to:

(C) $c_3 < c_1 < a_3 < a_1$.

A full-magic square M made up of distinct non-negative integers and satisfying (A) and (C) will be called a *reduced full-magic square.* Xin proves the following:

(III) *If M is a reduced full-magic square, then $a_2 = 0$.*

For the proof, note that since $b_1 + c_1 = a_2 + a_3$ and $b_3+c_3 = a_1+a_2$, we must have $a_2 < b_1$ and $a_2 < b_3$. Since $a_1 + a_2 + a_3 = c_1 + c_2 + c_3$, we have $a_2 < c_2$. Therefore, we have $a_2 < b_1, b_3, c_2$. So a_2 is the least side-middle element, and (by design) c_3 is the least corner element.

We cannot have $b_2 = 0$, as $b_2 = x > 0$. Since the square contains a 0, one of $\{a_2, c_3\}$ is 0. If $a_2 > 0$, then we get $c_3 = 0$. This implies that $a_1 = 2x$. Since $a_2 + a_3 = x$, we get $a_3 < x$; similarly, $c_1 < x$. But now we get $a_3 + b_2 + c_1 < 3x$, and we have lost the full-magic property. It follows that we cannot have $a_2 > 0$. Therefore, $a_2 = 0$ (and $c_3 > 0$).

Denote c_3 by y; then $y > 0$. The remaining entries can be found in terms of x and y by using the full-magic property, and we get the following:

$$M = \begin{pmatrix} 2x-y & 0 & x+y \\ 2y & x & 2x-2y \\ x-y & 2x & y \end{pmatrix} \qquad (2)$$

Since the elements of the square are distinct and non-negative, we have $x > y > 0$. Observing the matrix relation

$$\begin{pmatrix} 2x-y & 0 & x+y \\ 2y & x & 2x-2y \\ x-y & 2x & y \end{pmatrix} = x \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$+y \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix} \qquad (3)$$

we see that we have proved the following.

(IV) *Any reduced full-magic square M may be expressed as $M = x\,A + y\,B$, where x and y are positive integers with $x > y$, and*

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix} \qquad (4)$$

Further, the representation is unique, since A and B are linearly independent.

This may be termed as a *basis property of reduced full-magic squares.* Observe that the two basis squares obtained above have the full-magic property, but not the "distinct elements" property; further, B has some negative elements (and its magic sum is 0). Two examples of this basis representation are given below.

Original magic square	Reduced version	Representation
$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$	$\begin{pmatrix} 7 & 0 & 5 \\ 2 & 4 & 6 \\ 3 & 8 & 1 \end{pmatrix}$	$4A + B$
$\begin{pmatrix} 4 & 17 & 6 \\ 11 & 9 & 7 \\ 12 & 1 & 14 \end{pmatrix}$	$\begin{pmatrix} 13 & 0 & 11 \\ 6 & 8 & 10 \\ 5 & 16 & 3 \end{pmatrix}$	$8A + 3B$

Counting the Number of Reduced Full-Magic Squares with a Given Magic Sum

Consider equation (2) giving the form of a reduced full-magic square. If x is fixed, then the number of possible values for y is $x-1$, since $0 < y < x$. However, not all these values of y result in a square with distinct entries. For distinctness, the numbers

$$2x-y,\quad 0,\quad x+y,\quad 2y,\quad x,\quad 2x-2y,\quad x-y,\quad 2x,\quad y \tag{5}$$

must be distinct. We find that repeated values occur precisely when $x = 2y$, $x = 3y$ or $2x = 3y$. It follows that, for a given value of x,

Number of permissible values for

$$y = \begin{cases} x-1, & \text{if } x \equiv \pm 1 \pmod 6, \\ x-2, & \text{if } x \equiv \pm 2 \pmod 6, \\ x-3, & \text{if } x \equiv 3 \pmod 6, \\ x-4, & \text{if } x \equiv 0 \pmod 6. \end{cases} \tag{6}$$

The generating function that yields the number of reduced full-magic squares having a given magic sum $3x$ is therefore equal to

$$\sum_{x\equiv\pm1} (x-1)\ t^{3x} + \sum_{x\equiv\pm2} (x-2)\ t^{3x} +$$

$$\sum_{x\equiv3} (x-3)\ t^{3x} + \sum_{x\equiv0} (x-4)\ t^{3x} \tag{7}$$

The summations here are over $x \in \mathbb{N}$. (We have left out the "mod 6" from the summations for ease of typography.) To simplify this, it is best to use a computer algebra package like MATHEMATICA. This yields our next result.

(**IV**) *The number of reduced full-magic squares having a magic sum of S is the coefficient of t^S in the power series of the following rational expression:*

$$\frac{2\,(t^{12}+2\,t^{15})}{1-t^6-t^9+t^{15}}. \tag{8}$$

The power series is found to be

$$2\,t^{12}+4\,t^{15}+2\,t^{18}+6\,t^{21}+6\,t^{24}+6\,t^{27}+8\,t^{30}+10\,t^{33}+$$
$$8\,t^{36}+12\,t^{39}+12\,t^{42}+ \tag{9}$$

Remark 1. It is not likely that any such compact result exists for fourth order full-magic squares. For, using the set of numbers $\{0,1,2, \quad 7,8\}$, we can make just one third order full-magic square (up to rotations and reflections); whereas, using the set $\{0,1,2, \quad 13,14,15\}$, we can make 880 possible fourth order full-magic squares, not counting rotations and reflections!

Using the set of numbers {0, 1, ..., 7, 8}, we can make just 1 third-order full-magic square (up to rotations and reflections); whereas, using the set {0, 1, ..., 14, 15}, we can make 880 possible fourth-order full-magic squares, not counting rotations and reflections!

Semi-Magic Squares

As noted earlier, a square matrix is said to have the semi-magic property if its row sums and column sums are all equal to one another (there is no restriction on the diagonal sums); the common value of the sum is the "magic sum" of the matrix. The set of full-magic squares is obviously a subset of the set of semi-magic squares.

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Eigenvectors and Eigenvalues of a Semi-Magic Square

Observe that the vector $\mathbf{x}$ defined by

$$\mathbf{x} \;=\; (1,1,1)^T \tag{10}$$

is an eigenvector of any third order semi-magic square M the associated eigenvalue being S_M; for, if M is such a square, then $M\cdot\mathbf{x} = S_M\,\mathbf{x}$. This property allows us to

The product of two semi-magic squares is a semi-magic square, and its magic sum is the product of the magic sums of its two factors.

define a third order semi-magic square as one that has $\mathbf{x}$ as an eigenvector.

An easy corollary to this observation is that *the product of two semi-magic squares is a semi-magic square, and its magic sum is the product of the magic sums of its two factors*; for if M and N are two such squares, then

$$(MN)\ \mathbf{x} = M\ (N\ \mathbf{x}) = M\ (S_N\mathbf{x}) = S_M S_N \mathbf{x}. \tag{11}$$

In turn, it follows that the product of two full-magic squares has the semi-magic property. However, it may fail to have the full-magic property. For example, for the square P shown below,

$$P = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix} \tag{12}$$

we get:

$$P^2 = \begin{pmatrix} 64 & 40 & 40 \\ 40 & 64 & 40 \\ 40 & 40 & 64 \end{pmatrix} \qquad P^3 = \begin{pmatrix} 648 & 480 & 600 \\ 528 & 576 & 624 \\ 552 & 672 & 504 \end{pmatrix}$$

We see that P^2 has the semi-magic property (its magic sum is $144 = 12^2$) but not the full-magic property; on the other hand, P^3 does have the full-magic property, with magic sum $1728 = 12^3$. This obviously calls for an explanation; we shall provide one, later, using a different approach.

If P is a full-magic square, then P^2 is only a semi-magic square, but P^3 has the full-magic property.

Another easy corollary is that *if the inverse of a third order semi-magic square M exists, then it is a semi-magic square.* To see why, first note that if M^{-1} exists (equivalently, $\det(M) \neq 0$), then S_M is non-zero: for, we have $M\ \mathbf{x} = S_M\mathbf{x}$, therefore $\det(M - S_M I) = 0$; so if $S_M = 0$, then we get $\det(M) = 0$, contrary to hypothesis. But now the relation

$$M^{-1}\ \mathbf{x} = \frac{1}{S_M}\mathbf{x}, \tag{13}$$

shows that M^{-1} has the semi-magic property with magic sum equal to $1/S_M$.

For the full-magic square P in equation (12) we get:

$$P = \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{360} \begin{pmatrix} 53 & -52 & 23 \\ -22 & 8 & 38 \\ -7 & 68 & -37 \end{pmatrix},$$

Observe that P^{-1} has the full-magic property; its magic sum is $24/360 = 1/15$, and this is the reciprocal of the magic sum of P.

Remark 2. Semi-magic squares with zero determinant and non-zero magic sum do exist; for example, the following square, with magic sum 7 (its eigenvalues are -5, 0 and 7):

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 1 & 5 \\ 4 & 4 & -1 \end{pmatrix}.$$

If the inverse of a third order semi-magic square exists, then it is a semi-magic square.

Another Basis Property of Full-Magic Squares

We now prove a simple but extremely useful basis property of third order full-magic squares, and then use it to prove some nice properties of such squares.

Let the central element of a full-magic square M be x; then its magic sum is $3x$. Denote M_{11} by a and M_{12} by b. Making use of the full-magic property repeatedly, we get

$$M = \begin{pmatrix} a & b & 3x - a - b \\ 4x - 2a - b & x & 2a + b - 2x \\ a + b - x & 2x - b & 2x - a \end{pmatrix}. \qquad (14)$$

We see immediately that this may be written as

$$M = a\,U + b\,V + x\,W, \qquad (15)$$

If the inverse of a third order full-magic square exists, then it is a full-magic square; and its magic sum is the reciprocal of the magic sum of the original square.

where

$$U = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$W = \begin{pmatrix} 0 & 0 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 2 \end{pmatrix} \tag{16}$$

The matrices U V W are linearly independent, for if M is the zero matrix, then $a = b = x = 0$. So we arrive at another basis representation theorem.

Computing all possible two-term products of these matrices, we find that the nine products

$$U\ U, \quad U\ V, \quad U\ W, \quad V\ U, \quad V\ V, \quad V\ W, \quad W\ U,$$
$$W\ V, \quad W\ W,$$

all have the semi-magic property, but only U U has the full-magic property (in fact, U U is the zero matrix). For example, we find that

$$U{\cdot}V = \begin{pmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{pmatrix} \qquad U{\cdot}W = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$V\ V = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Observe that the diagonal sums differ from the row and column sums in all three products. Of the nine products, only W W has a non-zero magic sum. (This is not surprising, as U and V have zero magic sums.)

The product of two full-magic squares has the semi-magic property. However, it may fail to have the full-magic property.

It follows that *the product of two full-magic squares certainly has the semi-magic property, but in general it will fail to have the full-magic property.*

On the other hand, if we compute all possible three-term products of U, V W we find, to our great surprise, that *all twenty-seven products have the full-magic property*! This may be verified by the reader, using routine but (very) tedious computations. (The author did so using his ever-faithful MATHEMATICA.) For example, we have

$$U \quad V \quad W = \begin{pmatrix} 9 & 0 & -9 \\ -18 & 0 & 18 \\ 9 & 0 & -9 \end{pmatrix}$$

$$W \quad W \quad W = \begin{pmatrix} 18 & 18 & -9 \\ -18 & 9 & 36 \\ 27 & 0 & 0 \end{pmatrix}$$

We deduce from this that *the product of any three third order full-magic squares has the full-magic property.* It follows, inductively, that *the product of any odd number of full-magic squares has the full-magic property*, and that *if M is a full-magic square of order 3, then M^k has the full-magic property for any odd positive integer* k; for we have:

$$M^3 = M \quad M \quad M, \qquad M^5 = M^3 \quad M \quad M,$$

$$M^7 = M^5 \quad M \quad M, \qquad M^9 = M^7 \quad M \quad M, \qquad (17)$$

The product of any odd number of full-magic squares has the full-magic property; if M is a full-magic square of order 3, then M^k has the full-magic property for any odd positive integer k.

Lastly, we show that *if a third order full-magic square has an inverse, then the inverse is a full-magic square.* But this we do only by direct computation, so the proof cannot be called pretty! We find that if

$$M = \begin{pmatrix} a & b & 3x-a-b \\ 4x-2a-b & x & 2a+b-2x \\ a+b-x & 2x-b & 2x-a \end{pmatrix}$$

then $\det(M) = 9x\,(2ab + b^2 - 2ax - 4bx + 3x^2) = 9(2a + b - 3x)(b-x)x$; and if $\det(M) \neq 0$, then

$$\det(M)M^{-1} = \left(2ab + b^2 - 5ax - 4bx + 6x^2\right) U$$
$$+ \left(2ab + b^2 - 2ax - 7bx + 6x^2\right) V$$
$$+ \left(2ab + b^2 - 2ax - 4bx + 3x^2\right) W. \qquad (18)$$

It follows that the matrix $\det(M)\, M^{-1}$ has the full-magic property, with magic sum equal to

$$3\left(2ab + b^2 - 2ax - 4bx + 3x^2\right)$$

Naturally, M^{-1} too has the full-magic property.

Remark 3. The above proofs made repeated use of basis representations. It is not clear whether proofs can be manufactured that exploit the concepts of eigenvalue, trace, etc., as there is no obvious linear-algebraic significance that can be given to the sum of the "reverse main diagonal" of a square matrix.

"Magic Squares Indeed!"

The following property, first reported in [1], seems baffling at first encounter, but the basis property helps understand it. We illustrate it using the square P given in equation (12). Concatenating the entries in the columns to form the three-digit numbers 834, 159 and 672, then doing the same in the reverse order (and getting the numbers 438, 951 and 276), we come upon the following eyebrow-raising equality:

$$834^2 + 159^2 + 672^2 \;=\; 438^2 + 951^2 + 276^2$$

A similar equality holds if we work on the rows. The mystery may be resolved by observing that we actually have a *polynomial equality* hiding here:

$$(8z^2 + 3z + 4)^2 + (z^2 + 5z + 9)^2 + (6z^2 + 7z + 2)^2 =$$

$$(4z^2 + 3z + 8)^2 + (9z^2 + 5z + 1)^2 + (2z^2 + 7z + 6)^2,$$

both sides being equal to $101z^4+142z^3+189z^2+142z+101$. The substitution $z=10$ now yields the desired result. Now we must explain why the polynomial equality is true.

Let

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{19}$$

be a full-magic square, and let the rows and columns of M be denoted respectively by R_1, R_2, R_3, C_1, C_2, C_3 (each regarded as a vector). We must show that

$$\begin{aligned} R_1 \cdot R_1 = R_3 \cdot R_3, \quad R_1 \cdot R_2 = R_2 \cdot R_3, \\ C_1 \cdot C_1 = C_3 \cdot C_3, \quad C_1 \cdot C_2 = C_2 \cdot C_3, \end{aligned} \tag{20}$$

as this is what would make the polynomial equality true. Using the general form of the third order full-magic square given in equation (14), this becomes a routine verification. Let

$$M = \begin{pmatrix} a & b & 3x-a-b \\ 4x-2a-b & x & 2a+b-2x \\ a+b-x & 2x-b & 2x-a \end{pmatrix}$$

Then we readily find the following:

$$\begin{aligned} R_1 \cdot R_1 &= 2a^2+2ab+2b^2-6ax-6bx+9x^2 = R_3 \cdot R_3, \\ C_1 \cdot C_1 &= 6a^2+6ab+2b^2-18ax-10bx+17x^2 = C_3 \cdot C_3, \\ R_1 \cdot R_2 &= -4a^2-4ab-b^2+12ax+6bx-6x^2 = R_2 \cdot R_3, \\ C_1 \cdot C_2 &= -b^2+2bx+2x^2 = C_2 \cdot C_3. \end{aligned}$$

The mystery is now cleared.

Suggested Reading

[1] A Benjamin and K Yasuda, Magic Squares Indeed!, *American Mathematical Monthly*, February 1999.

[2] T K S Iyer, Magic Squares of Order Three, *Resonance*, Vol.11, No.9, pp.76-78, 2006.

[3] Guoce Xin, Constructing All Magic Squares of Order 3, available on the Web at the following URL: people.brandeis.edu/~maxima/files/papers/magicsquare.pdf