In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Magic Squares of Order Three

A magic square of order 3 is a 3×3 square in which the cells contain distinct non-negative integers such that the sum of the elements in the three rows, the three columns and the two main diagonals are all the same. This is called the *magic* property. Let the common sum (also called the "magic sum") be S, and let the number in the central square be x. We shall prove the following:

(I) The quantities S and x are related thus: $S = 3x$.

(II) For any given collection of nine distinct non-negative integers in arithmetic progression, there is "essentially" just one way of arranging them to make a magic square. If rotations and reflections are taken into account, then there are precisely eight ways.

Here is an example of such a square with $x = 9$ and $S = 27$:

4	17	6
11	9	7
12	1	14

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For the proofs, we denote the magic square as shown on the right:

a_1	a_2	a_3
b_1	b_2	b_3
c_1	c_2	c_3

Proof of (I)

The magic property gives us the following relations (one for each "line" passing through the central square):

$$S = a_1 + b_2 + c_3, \qquad S = a_2 + b_2 + c_2,$$

$$S = a_3 + b_2 + c_1, \qquad S = b_1 + b_2 + b_3.$$

By addition, we get $4S = (a_1 + a_2 + a_3 + b_1 + b_2 + b_3 + c_1 + c_2 + c_3) + 3b_2 = 3S + 3x$, from which follows $S = 3x$.

Proof of (II)

Without loss of generality, we may take the numbers to be 0, 1, 2, 3, 4, 5, 6, 7 and 8; for, if the least number in the arithmetic progression is a, and the common difference is d, then we may subtract a from the numbers, and then divide the resulting numbers by d. The "magic property" stays unchanged through these transformations, and the numbers obtained in the end are 0, 1, 2, 3, 4, 5, 6, 7 and 8.

As the sum of these nine numbers is 36, the magic sum of this square is $S = 36/3 = 12$. It follows from (I) that the central number is $x = 4$. So, for each line of the array that passes through the central square, the numbers in the two ends must add up to 8.

For a magic square of order 3 made up of distinct non-negative integers, the common sum ('magic sum') equals 3 times the number in the central cell.

Consider the placement of 0. If it belonged to a corner square, then it would belong to three different lines whose sum is 12 (a row, a column and a diagonal). Leaving aside the 0, the other two numbers in each of these lines would have to have a sum of 12. But there are only *two* ways of achieving a sum of 12 with the remaining numbers, namely, $4 + 8$ and $5 + 7$; the sum $6 + 6$ is not admissible, as a number cannot be repeated. So 0 cannot belong to a corner square. Accordingly, it must lie in the middle of some side. Let us place it in the middle

7	0	5
2	4	6
3	8	1

of the top row. The top row must now read $7+0+5$ or $5+0+7$. Opting for the first possibility, we now readily complete the magic square; we get the square shown on the left:

We could instead have opted for $5 + 0 + 7$ in the top row; in this case, we simply get a reflection of the above configuration. Or we could put 0 in the middle of some side other than the top row. Tracing through these different possibilities, we get eight configurations which are all rotations and reflections of the configuration given above. This justifies the claim made.

The 'S' Pattern

In the magic square above, the cells containing the middle five numbers (2, 3, 4, 5 and 6) are seen to form a sharply angled 'S' shape. This pattern may be put to use. For example, to construct a magic square using the consecutive even numbers 6, 8, 10, 20, 22, we start by entering the middle five numbers (10, 12, 14, 16, 18) in the 'S' shape; we get the array shown below (left). Since the magic sum for the square is $3 \times 14 = 42$, we readily get the remaining entries. The result is shown below (right).

For a given set of 9 distinct non-negative integers in arithmetic progression, there is just 1 way of arranging them to make a full-magic square, not counting rotations and reflections. If these are taken into account, then there are precisely 8 ways.

		16
10	14	18
12		

20	6	16
10	14	18
12	22	8

This method will clearly work for any given set of nine numbers in arithmetic progression. In passing, I note that when I asked my grand-daughter Shivaranjani (in standard 5) to make a 3×3 magic square with a set of consecutive numbers, she found the 'S' shape on her own and used it to complete the magic square.