Special Relativity, Causality and Quantum Mechanics – 2

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We discuss the aspects of non-locality and non-separability of states of composite quantum systems. The probabilistic nature of quantum mechanical predictions, and the impossibility of cloning, are shown to safeguard causality in the quantum world. An example of a game which can be won exploiting quantum entanglement, but which can never be won classically, is described.

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Peaceful Coexistence of Special Relativity and Quantum Mechanics

As discussed in Part 1, in the framework of the special theory of relativity, causality holds. This can be stated as follows: there is a finite speed for any signal, *i.e.*, for anything that carries information, and the highest speed for any signal is identical to the speed of light in vacuum. Einstein wanted his second postulate to transcend any particular physical theory. Now with the emergence of quantum mechanics, there is a nice chance to verify the second postulate or its consequence, causality.

Due to the existence of entangled states in quantum mechanics and its collapse postulate, there arose a big debate regarding the non-local features of quantum mechanics. In this case also Einstein was the pioneer, and along with Podolski and Rosen he showed that quantum mechanics is incomplete based on the assumptions that any physical theory must respect reality and locality. One should note that their reality criterion is more general than that of quantum mechanics, where reality can be assigned to an observable only when the quantum system is in one of the eigenstates of that observable.

Though quantum mechanics is incompatible with local realism, it does not allow any violation of causality as quantum mechanics is irreducibly probabilistic in nature.

However, they did not suggest any experiment to verify whether quantum mechanics is incompatible with a local realistic theory. It was shown in 1964 by Bell that all the results of quantum mechanics cannot be reproduced by a local realistic theory. From then on, some sort of non-locality has been assigned to quantum mechanics.

But interestingly, though quantum mechanics is incompatible with local realism, it does not allow any violation of causality as quantum mechanics is irreducibly probabilistic in nature.

Causality and Quantum Cloning

Here we shall concentrate our discussion on how the principle of causality can prohibit some non-physical operations in the quantum world. One such operation is cloning. Cloning of an unknown quantum state is prohibited in quantum mechanics by its mathematical formulation itself. But let us first observe how the existence of cloning in the quantum world could have implied the violation of causality.

Consider a state shared by Alice and Bob

$$|\psi\rangle_{sing} = \frac{1}{\sqrt{2}}[|\psi_z\rangle|\psi_{-z}\rangle - |\psi_{-z}\rangle|\psi_z\rangle]$$

where $|\psi_z\rangle$ and $|\psi_{-z}\rangle$ are spin up and spin down states of the particle along the z-direction. The state $|\psi\rangle_{sing}$ has the nice feature that it can be written in any basis while retaining a similar form. We write it in the x-basis,

$$|\psi\rangle_{sing} = \frac{1}{\sqrt{2}}[|\psi_x\rangle|\psi_{-x}\rangle - |\psi_{-x}\rangle|\psi_x\rangle]$$

If Alice measures in the z-basis, Bob's final state instantaneously jumps to either $|\psi_z\rangle$ or $|\psi_{-z}\rangle$. Similarly if Alice measures in the x-basis, then Bob's final state is either $|\psi_x\rangle$ or $|\psi_{-x}\rangle$. Now, if Bob has a way to learn the state of his particle, he could determine in which basis,

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either z-basis or x-basis, Alice has performed her measurement on her particle. But there is no way to learn the information about which state occurs out of these four states, namely, $|\psi_z\rangle$, $|\psi_{-z}\rangle$, $|\psi_x\rangle$ and $|\psi_{-x}\rangle$. The dimension of the Hilbert space corresponding to a spin-1/2 system being two, these states are linearly dependent. In quantum mechanics there is no measurement strategy which can reliably distinguish between linearly dependent vectors. Had cloning been possible, Bob could have applied the cloning machine repeatedly (n times, say) on his particle to clone the unknown state. Then the possible state on Bob's side will be one of the four vectors $|\psi_z\rangle^{\otimes n}$ (n copies of $|\psi_z\rangle$), $|\psi_{-z}\rangle^{\otimes n}$, $|\psi_x\rangle^{\otimes n}$ and $|\psi_{-x}\rangle^{\otimes n}$. Interestingly, these four vectors in 2^n dimensional Hilbert space are linearly independent, and they are almost orthogonal for large n. So Bob can determine the state and hence can find out the basis in which Alice has measured her particles.

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Now consider a situation where Alice and her brother Bob are light years away, and they share a singlet state. They have earlier decided a fixed date on which Alice has to inform Bob within 24 hours whether their father who is staying with Alice is alive or dead. They have also agreed that in case their father is alive, Alice will measure in the z-basis, and in the x-basis if he is dead. Let Alice perform a measurement on her particle in one of the two bases depending on the condition of her father at 6 in the morning on the date earlier fixed; Bob will try to find out the information some time in the evening on the same date. Now, in the presence of a cloning machine, the protocol described above will enable Bob to learn about his father within 24 hours. But this would imply that the speed of real information transfer is much greater than that of light. Hence another inertial frame can be chosen where Bob's action (effect) will precede Alice's action (cause). Then it will be an example of a violation of causality.

General Quantum Operation and Causality

In general there is no quantum mechanical operation by which one can violate the principle of causality. In quantum mechanics, the dynamics of a closed system is described by a unitary transformation which is expressed by the Schrodinger equation. Then in a natural way the dynamics of an open system should be described by a unitary transformation on the system of interest and its environment which together again form a closed system. Now if we are interested in an operation on the system alone, we have to ignore the environment after the unitary transformation which, mathematically, means performing a partial trace over the environment to obtain the reduced state of the system alone.

$$T(\rho_{sys}) = Tr_{env}[U(\rho_{sys} \otimes \rho_{env})U^{\dagger}].$$

All the physical operations on the system can be obtained in the above way; mathematically, they are characterized by linear completely positive (CP) maps. A map T is completely positive if T maps (by acting on one system A) every density matrix (which is by definition, a positive operator) defined on a joint system A and B to another valid density matrix, whatever be the dimension of Hilbert space corresponding to the system B. According to the Kraus representation theorem, any trace preserving CP map, *i.e.*, a general quantum mechanical operation, can also be expressed in the following form

$$T(\rho) = \sum_k A_k \rho A_k^{\dagger},$$

where A_k is some set of operators with $\sum_k A_k^{\dagger} A_k = I$ (unit operator), which is very useful. This result is at least as old as the 1961 work of E C G Sudarshan, P M Mathews, and J Rau (*Phys. Rev.* Vol.121, p.920, 1961, Theorem 3) but it has picked up a nickname in the literature.

Now consider a situation where Alice and Bob share a general entangled state ρ_{AB}. The density matrix of Alice system is given by $\rho_A = Tr_B(\rho_{AB})$. Now we assume that Bob has performed a general operation (trace preserving) on his subsystem of the composite state ρ_{AB}. The final state of the composite system becomes

$$\rho'_{AB} = T'(\rho_{AB}) = \sum_k I \otimes A_k(\rho_{AB}) I \otimes A_k^\dagger.$$

The state of Alice's system after Bob's operation is given by

$$\rho'_A = Tr_B(\rho'_{AB}) = Tr_B(\sum_k I \otimes A_k(\rho_{AB}) I \otimes A_k^\dagger)$$

$$= Tr_B(\sum_k I \otimes A_k^\dagger A_k(\rho_{AB})) = Tr_B(\rho_{AB}) = \rho_A,$$

i.e., by means of any operation on his system, Bob cannot change the reduced density matrix on Alice's side. So the principle of causality will always be maintained in this scenario.

Can the allowed physical operations in quantum mechanics i.e., completely positive maps, be derived from the principle of causality?

We now consider the reverse question: can the allowed physical operations in quantum mechanics *i.e.*, completely positive maps, be derived from the principle of causality? In 2001, Simon *et al* [1] showed that the fact that the most general dynamics has to be described by linear completely positive maps on density matrices can be derived from the following assumptions:

1. Physical states are described by rays in Hilbert space.
2. The probabilities of the outcomes of a measurement are calculated using the Born rule.
3. Superluminal communication is excluded.

One of the peculiar features of quantum mechanics is that a mixed state can be prepared in many different ways. For example, an equal mixture of spin up and spin down states of a spin 1/2 particle along the z direction and a similar mixture along the x direction give rise to the same density matrix. So a general density matrix ρ (say) can be written in many representations:

$$\rho = \sum p_i|\psi_i\rangle\langle\psi_i| = \sum q_j|\phi_j\rangle\langle\phi_j| = ...$$

Now there is a celebrated theorem, namely, the GHJW theorem (see *Box* 1) which says that every probabilistic mixture of pure states corresponding to the same density matrix ρ_A can be remotely prepared via an appropriate measurement by B if a joint system A and B is prepared in a pure entangled state $|\psi_{AB}\rangle$, such that $\rho_A = Tr_B|\psi_{AB}\rangle\langle\psi_{AB}|$.

Consider a general dynamical evolution in system A of the form g: $|\psi\rangle\langle\psi| \rightarrow g(|\psi\rangle\langle\psi|)$, where g is not necessarily linear. Under such a dynamics, the probabilistic mixture $\{|\psi_k\rangle\langle\psi_k|, p_k\}$ goes to another probabilistic mixture $\{g(|\psi_k\rangle\langle\psi_k|), p_k\}$. Therefore the two final density matrices obtained after the action of g on two different probabilistic mixtures $\{|\psi_i\rangle\langle\psi_i|, p_i\}$ and $\{|\phi_j\rangle\langle\phi_j|, q_j\}$ will take the forms

$$\begin{aligned} \rho'_A(\{|\psi_i\rangle\langle\psi_i|), p_i\}) &= \textstyle\sum_i p_i g(|\psi_i\rangle\langle\psi_i|), \\ \rho'_A(\{|\phi_j\rangle\langle\phi_j|), q_j\}) &= \textstyle\sum_j q_j g(|\phi_j\rangle\langle\phi_j|), \end{aligned} \tag{1}$$

which are not assumed *a priori* to be equal although both the ensembles in the input correspond to same density matrix. According to the second assumption, the results of all standard quantum measurements in A at a given time are determined by the reduced density matrix ρ'_A. Now if the above two final density matrices are not equal, then Alice can use this g operation to learn which representation Bob has prepared in her system by acting (performing a unitary operation and then a measurement in an appropriate basis) remotely on his system if Alice knows the initial entangled state. But that would imply signalling or a violation of causality.

Box 1. Proof of the GHJW Theorem

Consider the spectral representation of the density matrix ρ_A, *i.e.*,

$$\rho_A = \sum_i p_i |\eta_i\rangle\langle\eta_i|, \qquad \sum p_i = 1,$$

where the $\{|\eta_i\rangle's\}$ are orthogonal to each other. ρ_A can be realized as an ensemble in which each pure state $|\eta_i\rangle\langle\eta_i|$ occurs with the probability p_i. If ρ_A is not degenerate, then its spectral representation is also unique. Now Bob can remotely prepare this ensemble in the following way. Suppose that Alice and Bob share a bipartite pure entangled state

$$|\phi_1\rangle_{AB} = \sum_i \sqrt{p_i}|\eta_i\rangle_A \otimes |\alpha_i\rangle_B,$$

where the vectors $|\alpha_i\rangle_B \in H_B$ are mutually orthogonal and normalized. Now a measurement in the $|\alpha_i\rangle_B$ basis in system B will prepare the density matrix $\rho_A = \sum_i p_i|\eta_i\rangle\langle\eta_i|$ for the system A. $|\phi_1\rangle_{AB}$ is called a purification of ρ_A.

Consider any other probabilistic mixture of the same density matrix

$$\rho_A = \sum_\mu q_\mu|\psi_\mu\rangle\langle\psi_\mu|, \qquad \sum q_\mu = 1,$$

where the $\{|\psi_\mu\rangle's\}$ are not orthogonal in general. Then one can have the corresponding purification for this ensemble

$$|\phi_2\rangle_{AB} = \sum \sqrt{q_\mu}|\psi_\mu\rangle_A \otimes |\beta_\mu\rangle_B,$$

where again $\{|\beta_\mu\rangle_B's\}$ are orthonormal vectors in H_B. Again by performing an orthogonal measurement in the $|\beta_\mu\rangle_B$ basis in the B system, the above ensemble can be prepared for system A.

But the Schmidt decomposition of $|\phi_2\rangle_{AB}$ must have the form

$$|\phi_2\rangle_{AB} = \sum \sqrt{p_i}|\eta_i\rangle_A \otimes |\xi_i\rangle_B,$$

where the vectors $|\xi_i\rangle_B \in H_B$ are mutually orthogonal and normalized, as it has to reproduce the correct reduced density matrix for A. Let us now define a unitary operator U such that $U|\xi_i\rangle = |\alpha_i\rangle$: then

$$|\phi_1\rangle_{AB} = I \otimes U|\phi_2\rangle_{AB}.$$

Thus we have seen that starting from a single purification ($|\phi_1\rangle_{AB}$, say), Bob can prepare any representation leading to the same density matrix for Alice by the proper choice of a unitary operator and a measurement basis.

So both the final density matrices can only be a function of the original density matrix, not of its particular representation.

$$\rho'_A = g(\rho_A) = \sum_i p_i g(|\psi_i\rangle\langle\psi_i|) = \sum_j q_j g(|\phi_j\rangle\langle\phi_j|) \quad (2)$$

i.e., g now extends to density matrices. Obviously, the operation g acts in the same way for any representation of ρ_A, and it implies that g is linear. For any representation of a mixed state, g works in the following way;

$$g(\sum p_i|\psi_i\rangle\langle\psi_i|) = \sum_i p_i g(|\psi_i\rangle\langle\psi_i|).$$

Following the second assumption, g is positive (g maps a positive operator to a positive operator) to ensure that $g(\rho_A)$ is again a valid density matrix. Thus we have seen that g, a general dynamical evolution, has to be linear and positive under the above three assumptions. But we have to show that the map g is completely positive: completely positive simply means that if system A evolves and system B does not, any initial density matrix of the combined system evolves to another density matrix. We assume, on the contrary, that g is not completely positive. Then there must be an entangled state ρ_{AB} of the combined system, such that $g \otimes I(\rho_{AB})$ is not positive, which means the physical operation g takes a density operator ρ_{AB} to a non-density operator by acting on the subsystem A of a joint state ρ_{AB}. But a negative operator would produce negative probabilities contradicting the second assumption. So the existence of entangled states for the joint system forces the operation g to be completely positive. So the first two assumptions, peculiar to the quantum world, along with causality are sufficient to derive the dynamical rule of quantum physics.

Quantum Non-Locality and Causality

As we have discussed, the causality constraint does not disallow a physical theory which violates local realism. Quantum mechanics is one such physical theory. But does this kind of non-locality have any operational role? Apparently it seems that quantum entanglement is useless as the non-locality anyway has to respect causality. But this is wrong. It can perform something which would seem like a miracle in the classical world. Here we describe a game played by three players far away from each other; the game is such that it cannot be won in a classical world without communication [2]. But in a quantum world, the players can win it without any communication. This shows the subtlety of quantum non-locality; though it respects causality. it still performs something which in a classical world would seem like some sort of signalling.

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Description of the Game

Let a team of three players, Alice, Bob and Charlie be situated far apart from each other. Before being far apart they are allowed to make any preparation sharing a classical system, or a quantum system, or both. Then at a certain time t, each player is asked one of the two possible questions: "What is X?" or "What is Y?" Each player will have to give an answer which is limited to only two possibilities: "1" or "–1" for both the questions.

Of course, the questions that would be placed before them in each turn are not completely random. According to the rules of the game, either all players are asked the X question, or only one player is asked the X question and the other two are asked the Y question. The game will be repeated many times. The team wins if in each turn the product of their three answers is -1 in the case of three X questions, and is 1 in the case of one X and two Y questions. The players are completely free to choose their answers either randomly or using some protocol based on their correlated systems prepared before the game. The only constraint is that no classical communication is allowed among them.

Classical Protocol

Since no player is allowed to get any message about the questions asked of the other players, an optimal strategy should correspond to some prior definite decisions of each player: which answer to give for the different possible questions. But it is easy to prove that any such strategy cannot ensure a win for all allowed combinations of questions. On the contrary, we assume that there is a set of answers $\{X_A, Y_A, X_B, Y_B, X_C, Y_C\}$, where X_A is the answer of Alice to question X, etc., which satisfy the winning conditions of the game for all possible questions. Then the answer set has to satisfy the following equations,

$$\begin{aligned} X_A X_B X_C &= -1, \\ X_A Y_B Y_C &= 1, \\ Y_A X_B Y_C &= 1, \\ Y_A Y_B X_C &= 1. \end{aligned} \tag{3}$$

Note that X_A, the answer for question X given by Alice, has appeared in the first and second equations, and in both the equations it will have the same value (1 or –1) since it cannot depend on the questions given to the other players as they will not be allowed to communicate. The same is true for the other answers.

The four equations in (3), however, are incompatible, because the product of all the left hand sides of equation (3) is the product of squares of numbers which are ± 1 and therefore it equals 1, while the product of all the right hand sides of these equations is -1. This contradiction proves that there cannot be any classical strategy to win the game.

Quantum Winning Strategy

The solution provided by quantum theory is as follows. Suppose that Alice, Bob and Charlie share the three qubit GHZ state [3]

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|1\rangle_C).$$

Their strategy to win the game is this:

1. If a member of the team is asked the X question, he/she will measure the spin of his/her qubit along x-direction *i.e.*, he/she will measure σ_x, otherwise he/she will measure σ_y (spin observable along y-direction). Then according to the rules of the game, the corresponding operators will be

$$\sigma_x^A\sigma_x^B\sigma_x^C \quad \sigma_x^A\sigma_y^B\sigma_y^C \quad \sigma_y^A\sigma_x^B\sigma_y^C \quad \sigma_y^A\sigma_y^B\sigma_x^C$$

2. The results of the measurements (+1 or −1 eigenvalues) will be their answers.

It is interesting to note that for the first observable, $\sigma_x^A\sigma_x^B\sigma_x^C$ $|\psi\rangle_{GHZ}$ is an eigenstate with eigenvalue −1, and for the last three observables, $\sigma_x^A\sigma_y^B\sigma_y^C$ $\sigma_y^A\sigma_x^B\sigma_y^C$ $\sigma_y^A\sigma_y^B\sigma_x^C$ $|\psi\rangle_{GHZ}$ is an eigenstate with eigenvalue +1, *i.e.*,

$$\sigma_x^A\sigma_x^B\sigma_x^C|\psi\rangle_{GHZ} = -1|\psi\rangle_{GHZ} \quad (4)$$

$$\sigma_x^A\sigma_y^B\sigma_y^C|\psi\rangle_{GHZ} = +1|\psi\rangle_{GHZ} \quad (5)$$

$$\sigma_y^A\sigma_x^B\sigma_y^C|\psi\rangle_{GHZ} = +1|\psi\rangle_{GHZ} \quad (6)$$

$$\sigma_y^A\sigma_y^B\sigma_x^C|\psi\rangle_{GHZ} = +1|\psi\rangle_{GHZ} \quad (7)$$

It is clear from the above equations that in every case they will win the game.

Now to make the situation more glaring, we can place the three players, say, $\frac{1}{365}$ light years away from each other. They are asked the questions at the same time, and they are asked to give the answers within one hour. Now even if they are allowed to communicate with each

Suggested Reading

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other or if we allow local actions of one to reach the other parties, even then, in a classical world, it is impossible to give the answers which will help them to win the game within the stipulated time. But the quantum protocol described above has nothing to do with all these time descriptions; in fact, no communication is necessary.

One should carefully observe that though in the quantum protocol, the game is won deterministically, the players do not decide in advance the answers they will give for each question. They get their answers from the results of their local measurements where the local measurement results are inherently random, but they are globally correlated via the GHZ state. This is the key behind the successful strategy described above. Obviously this quantum protocol does not allow the players to know the questions that have been put to the other players. In this way it respects causality.

Conclusion

Hypothetically, one can show that in a probabilistic theory, local correlations are not the only correlations which respect causality [4]. There can be peaceful coexistence of causality and some non-local correlations, *i.e.*, they are not logically contradictory. It is a surprising feature of nature that it allows some non-local correlations to exist within its overall constraint of causality, and quantum correlations are the only existing incarnation of that feature in the present day world.

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