Some Interesting Mathematical Gems

V G Tikekar

As the title of the article indicates, I am going to point out certain bright and beautiful pieces of mathematical work which can be treated as gems. Of course, choice of these pearl-like items is subjective depending on my understanding and appreciation (or lack of it) of the areas of mathematics in which I find interest.

V G Tikekar is a former professor of mathematics at IISc, Bangalore. His research interests are in mathematical programming, numerical mathematics, theoretical computer science, probability and statistics.

According to Bell [1], "It is difficult if not impossible to state why some theorems in arithmetic are considered 'important' while others equally difficult to prove are dubbed trivial. Same thing applies to results chosen here" I shall not refer to gems like Gödel's incomplete theorem or the prime number theorem, which occupy such a high place in mathematical literature that every student, teacher, and researcher of mathematics will surely be expected to know them. My endeavour here is to point out those gems or pearls which I feel you might have missed while carrying out your study of the canvas of mathematics which is so vast and widely spread in all directions of human activity. Again, the gems that I have selected are the creations of mathematicians, some of whom are well-known and some not so well-known.

Based on an invited talk dedicated to Dr P L Bhatnagar, Founder Professor of the Mathematics Department (then known as Department of Applied Mathematics) of IISc, at the Golden Jubilee Conference of the Department on 23rd March 2006.

—♦—

1. The first gem I select to present is Euler's remarkable observation. In his work he came across the infinite product

$$(1-x)(1-x^2)(1-x^3)(1-x^4) \qquad (1.1)$$

Keywords
Pentagonal numbers, Pik's formula, power triangle, formulae to generate prime numbers, Kaprekar's conjecture, Rogers–Ramanujan identity, hypersphere and hypercube.

The gems selected are the creations of mathematicians, some of whom are well-known and some not so well-known.

whose expansion he obviously obtained as

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\cdot \quad (1.2)$$

Normally, one might have carried out one's work with this product without observing the special property, of the indices of x in (1.2), which Euler noticed. The indices of x in the order of occurrence (noting $1 = x^\circ$) are:

$$0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \quad (1.3)$$

Euler observed and noted that the alternate indices

$$1, 5, 12, 22, 35, 51, \quad (1.4)$$

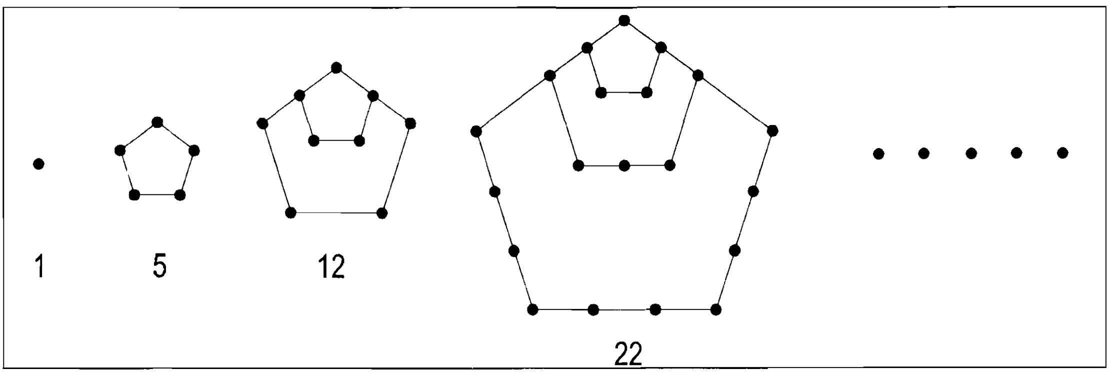
are generated by the formula

$$(1/2)n(3n-1), \quad n = 1, 2, 3, \quad (1.5)$$

These are known as pentagonal numbers, which are a particular case of the class of polygonal numbers. Pictorially (or geometrically) these pentagonal numbers are represented using regular pentagons (and hence, perhaps, their name) as shown in *Figure* 1.1.

It would perhaps be considered as 'not much' as it came from the all-time-great Euler. But Euler's brilliance did not stop there. He observed further (and observation is a very important tool and quality in the study of science)

Figure 1.1



that the remaining alternately occuring indices of x in (1.3), namely

$$0, 2, 7, 15, 26, 40, \tag{1.6}$$

are obtainable from the same formula (1.5) if we substitute $n = 0, -1, -2$, in it. These cannot be represented pictorially using pentagons and cannot be genuinely called pentagonal numbers. Of course we are at liberty to call them generalized pentagonal numbers, as they are generated by the same formula (1.5), which generates genuine pentagonal numbers (1.4), by putting zero, and negative values of n. This observation by Euler is indeed a masterpiece [2].

It is remarkable that Euler spotted the presence of pentagonal numbers in a serious work of mathematics.

In the serious study of mathematics recommended for undergraduate, postgraduate and doctoral students of mathematics, real, complex and prime numbers are included, but almost never polygonal numbers, which are normally considered a part of recreational mathematics. So it is remarkable that Euler brilliantly spotted the presence of pentagonal, and so-called generalized pentagonal numbers in a serious work of mathematics, showing his deep insight. Perhaps, this was not enough. The recurrence formula, also given by Euler, for the number, $p(n)$, of partitions of a natural number n is

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - p(n-22) + p(n-26) + p(n-35) + p(n-40) - p(n-51) - \tag{1.7}$$

It also contains the genuine and generalized pentagonal numbers, again occurring alternately in successive terms.

–♦–

2. The second gem I wish to point out is Pik's formula in Elementary Geometry. It was given by Vienna-born George Alexander Pik, to find the area of a polygon. Let

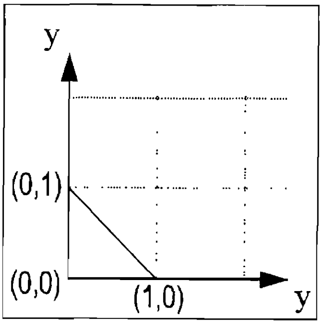


Figure 2.1.

us recall that in a coordinate plane, the points whose both coordinates are integers are called lattice points. In such a plane, the area of a polygon P is given by

$$\text{Area(P)} = i + \frac{b}{2} - 1, \tag{2.1}$$

where i = number of lattice points that lie completely inside P, b = number of lattice points that lie on the boundary of P. As an example, consider the triangle whose vertices are given by (0,0), (0.1) and (1,0) (see *Figure* 2.1). Then it is clear from the figure, that $i = 0$, and $b = 3$.

So, by Pik's formula

$$\text{Area}(\Delta \text{ in } Figure\ 2.1) = 0 + \frac{3}{2} - 1 = \frac{1}{2} \tag{2.2}$$

which can be verified to be correct by the usual formula (1/2) base × altitude. Similarly, look at *Figure* 2.2 showing a general shaped polygon ACDEFGHIJ. For this polygon, $i = 5$ (viz., I,J,K,L,M), and $b = 10$ (these lattice points are numbered 1 to 10 in the figure). So

$$\text{Area (polygon ACDEFGHIJ)} = 5 + \frac{10}{2} - 1 = 9. \tag{2.3}$$

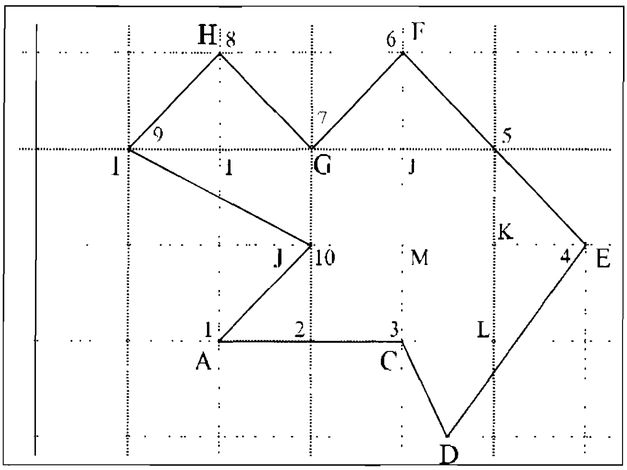


Figure 2.2.

Why I call Pik's formula a gem, is illustrated by the example associated with *Figure* 2.3. There ABC is such a triangle that its 3 vertices A,B,C are the only lattice points on its boundary, while D is the only lattice point inside it. In such a case D turns out to be the centroid (common point at which the 3 medians of a triangle meet) of ΔABC. To prove this, notice that the situation of each of the three triangles ABD, ACD, and BCD is exactly similar to the triangle in *Figure* 2.1. So

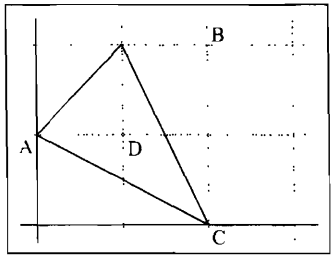


Figure 2.3.

$$\text{Area } (\Delta ABD) = \text{Area } (\Delta ACD) = \text{Area } (\Delta BCD) = 1/2 \quad (2.4)$$

From this result (obtained using Pik's formula), the proof of the proposition that D is the centroid of ΔABC follows in a straightforward fashion. If one just imagines the proof of the proposition without resort to Pik's formula, one would appreciate my calling Pik's formula, a gem.

–♦–

3. Consider the triangular arrangements of numbers, given in *Figures* 3.1(a-c). *Figure* 3.1(a), and *Figure* 3.1(b) are called arithmetic (or Pascal) and harmonic triangle respectively. *Figure* 3(c) has no special name. Each one of these three triangular arrangements has a variety of interesting properties and uses, which I shall not dwell upon here. They have been shown here only to say that there exists a fourth triangular arrangement, given in *Figure* 3.2. It is constructed using the following rules: (i) It has infinite number of rows, numbered 0,1,2,3,... starting from the top. (ii) It's nth row ($n = 0, 1, 2, \ \ .$) has $n + 1$ members, which are numbers. (iii) First member of each row is 1. (iv) Every member in each row has a suffix; (v) the suffix of the kth member ($k = 1, 2, 3, \ \ .$) in every row is k. (vi) A member at any place (except the first member which is governed by rule (iii) above) in a particular row = (member on the immediate left in the immediate upper row) $\times$ (its suffix)

Figure 3.1. (a) Arithmetic* Δ*; (b) Harmonic* Δ*; (c).

(a)

				1				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

(b)

				1				
			1/2		1/2			
		1/3		1/6		1/3		
	1/4		1/12		1/12		1/4	
1/5		1/20		1/30		1/20		1/5

(c)

				1				
			3		5			
		7		9		11		
	13		15		17		19	
21		23		25		27		29

+ (member on the immediate right in the immediate upper row)[1] × (its suffix).

[1] This will be zero if no such member exists, which happens when we compute the last member for a row.

This triangular arrangement is a recent discovery, compared to the origins of the three triangular arrangements of *Figure* 3.1.

The successive rows of the triangle of *Figure* 3.2 are useful as follows:

$$\left.\begin{array}{ll}
1^\circ + 2^\circ + 3^\circ + \quad + n^\circ &= 1\ {}^nC_1 \\
1^1 + 2^1 + 3^1 + \quad + n^1 &= 1\ {}^nC_1 + 1\ {}^nC_2 \\
1^2 + 2^2 + 3^2 + \quad + n^2 &= 1\ {}^nC_1 + 3\ {}^nC_2 + 2\ {}^nC_3 \\
1^3 + 2^3 + 3^3 + \quad + n^3 &= 1\ {}^nC_1 + 7\ {}^nC_2 + 12\ {}^nC_3 + \\
 & \qquad 6\ {}^nC_4 \\
1^4 + 2^4 + 3^4 + \quad + n^4 &= 1\ {}^nC_1 + 15\ {}^nC_2 + 50\ {}^nC_3 + \\
 & \qquad 60\ {}^nC_4 + 24\ {}^nC_5
\end{array}\right\} \qquad (3.1)$$

				1_1					$\longleftarrow$ 0th row
			1_1		1_2				$\longleftarrow$ 1st row
		1_1		3_2		2_3			$\longleftarrow$ 2nd row
	1_1		7_2		12_3		6_4		$\longleftarrow$ 3rd row
1_1		15_2		50_3		60_4		24_5	$\longleftarrow$ 4th row

Figure 3.2 (Power Δ).

We derive the formulae $\sum_{j=1}^{n} j = n(n+1)/2$, and $\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$ in the beginning of college-studies. But the formulae for $\sum_{j=1}^{n} j^k$ for any k were not known. They become easily available from this triangle. That is a great achievement. That is why I reckon it as a gem. Since this triangle gives the sum of kth powers ($k = 0, 1, 2, \ldots$) of first n natural numbers for any non-negative integer k, it is called 'power triangle' [3]. This result is very powerful also.

–♦–

4. Now I quote the following from [4]: "Even though the number of prime numbers is infinite, we can still tame them and handle them easily if we can find a formula which gives all prime numbers or at least a formula which gives only prime numbers. Many such formulae have been proposed, but all of them have been proved to be wrong" This is written by one of the brilliant mathematicians of India.The last statement in the quote above is not true as will be seen from the discussion that follows.

Similarly, in 2004 I listened to a lecture, in USA, by Dudley Woody, professor of mathematics at DuPan University, USA. He is known to be a very good expositer of mathematical topics. He gave, apart from many other formulae, the following ones to calculate p_n, the nth prime.

$$(i) \quad p_n = 1 + \sum_{m=1}^{2^n} \left\{ \left[\frac{n}{\sum_{j=1}^{n} F(j)} \right]^{1/n} \right\}$$

$$F(j) = \cos^2 \pi \frac{(j-1)! + 1}{j}, \quad j \in \mathbb{N}, \quad j \geq 1.$$

(due to Willan)

Note: $F(j) = 1$ if j is prime, and zero otherwise.

$$(ii) \quad p_n = 1 + \sum_{m=1}^{2^n} \left[\left(\frac{n}{1 + \pi(m)} \right)^{1/n} \right]$$

$$\pi(m) = \sum_{j=2}^{m} H(j), \quad m = 2, 3,$$

$$H(j) = \left[\sin^2 \pi \frac{\{(j-1)!\}^2}{j} \Big/ \sin^2(\pi/j) \right] \quad \text{(due to Willan)}$$

or

$$\pi(m) = \sum_{j=2}^{m} \left[\frac{(j-1)! + 1}{j} - \left\{ \frac{(j-1)!}{j} \right\} \right] \quad \text{(due to Minac)}$$

Here $\pi(m)$ is the prime counting function.

$$(iii) \quad p_n = \left[1 - \frac{1}{\log 2} \log \left(-\frac{1}{2} + \sum_{d|P_{n-1}} \frac{\mu(d)}{2^d - 1} \right) \right]$$

$$P_{n-1} = p_1 p_2 \quad \cdot p_{n-1} \quad \text{(due to Gandhi)}$$

and $\mu(d)$ is the Möbius function.

Woody also mentioned in his lecture a 26-variable polynomial of degree 25 which was discovered by Jones, Sato, Wada, and Wiens in 1976 and which generates all (but not only) primes [4]. Woody observed in his lecture, and you can also satisfy yourselves that the above-mentioned formulae are very cumbersome and not-so-good for producing primes. But, to my surprise and dismay Woody did not mention the formula which was in my mind, and which I consider a gem. So at the end of his lecture, I wrote on the board, with his permission, the following formula [5], which generates all primes, and every odd prime exactly once.

$$f(x,y) = \frac{y-1}{2}[|B^2-1| - (B^2-1)] + 2, \quad x, y \in \mathbb{N}$$

$$\text{where} \quad B = x(y+1) - (y!+1). \tag{4.1}$$

I also stated the only weakness of this formula, namely that it generates the prime 2 for many pairs (x, y). I asked Woody whether he would consider the formula (4.1) as a gem and he immediately answered, emphatically saying 'yes' This formula appears in the book [5] entitled *Mathematical Gems.*

–♦–

5. Consider the prime 11. Add to it its individual digits to get 11+1+1=13. Since 13 is also prime, do the same as was done to 11 to get 13+1+3=17. Again 17 being also a prime, repeat the procedure and we get 17+1+7=25. We stop here since 25 is a composite. Thus starting from the prime 11, we went to 13 in the first step. Second and third steps comprised going from 13 to 17, and going from 17 to 25, respectively. So to get the composite from the prime 11 by 'digit-addition' procedure, we require 3 steps. Can we talk about the maximum number of steps that would be required to get a composite from any prime by such a procedure? Some of the primes that need only 1 step to reach a composite are: 17, 31, 41, 109, 3001, 2003; some that require two steps are: 127, 307, 587, 1009, 1061, 1087; some that need three steps are 11, 101, 149, 167, 367, 479, 1409; and some of the primes in whose case 4 steps are needed are: 277, 1559.

37783	85601	259631	268721	350941	371939	378901
516521	665111	733331	883331	967781	1047929	1056521
1081721	1258811	1427411	1480573	1515929	1584901	1614929
1842131	1872311	1885981	2027801	2044873	2450531	2759111
2847991						

516493	1056493	1427383	1885943

Table 5.1 (top). The primes that need 5 steps to reach a prime.
Table 5.2(bottom). The primes that require 6 steps to reach a prime.

D R Kaprekar (1905-1986), a high-school teacher at Deolali, who has published his work in the journal *Recreational Number Theory, Scripta Mathematica* etc, and whose expertise in dealing with numbers was brought to the notice of the world, by Martin Gardner through *Scientific American* [6], made a conjecture, on the basis of his calculations, that the steps required to get a composite from a prime, employing the digit-addition procedure explained above, would be at most four [7].

One of my articles, in which I had mentioned this conjecture by Kaprekar, was translated into Gujarati. It was read by Vishal Joshi from Jamnagar. among others. In a letter dated November 29, 2001 and addressed to me, Vishal, a BSc (Physics) student then, disproved Kaprekar's conjecture by giving counter-examples. He enumerated 29 primes (see *Table* 5.1), each of them requiring 5 steps. He also gave 4 primes (see *Table* 5.2), each of which needs 6 steps.

D R Kaprekar, a high-school teacher at Deolali, whose expertise in dealing with numbers was brought to the notice of the world by Martin Gardner.

Any piece of creative work, getting judged for its importance and merit should be viewed from the angle of the background knowledge of the creator, apart from the intrinsic quality and worth of the work. The dedicated worker Kaprekar who was submerged in the world of numbers, did not have good appreciation of certain concepts like 'set' or 'proof' Taking into consideration the background of Kaprekar and Joshi, I would grant the status of a gem to the Kaprekar conjecture and its disproof.

–♦–

10 9+1 8+2 8+1+1 7+3 7+2+1 7+1+1+1 6+4 6+3+1 6+2+2 6+2+1+1
6+1+1+1+1 5+5 5+4+1 5+3+2 5+3+1+1 5+2+2+1 5+2+1+1+1
5+1+1+1+1+1 4+4+2 4+4+1+1 4+3+3 4+3+2+1 4+3+1+1+1 4+2+2+2
4+2+2+1+1 4+2+1+1+1+1 4+1+1+1+1+1+1 3+3+3+1 3+3+2+2 3+3+2+1+1
3+3+1+1+1+1 3+2+2+2+1 3+2+2+1+1+1 3+2+1+1+1+1+1 3+1+1+1+1+1+1+1
2+2+2+2+2 2+2+2+2+1+1 2+2+2+1+1+1+1 2+2+1+1+1+1+1+1 2+1+1+1+1+1+1+1+1
1+1+1+1+1+1+1+1+1+1

Table 6.1 All the 42 partitions of 10.

6. Many mathematical works of Ramanujan can be classified as gems. I cite here just one. Consider all the 42 partitions of 10. They are listed in *Table* 6.1 for ready reference. From them, you can identify only the following six having the property that in each of them, difference between any two parts of the partition is ≥ 2:

$$10, 9+1, 8+2, 7+3, 6+4, 6+3+1. \qquad (6.1)$$

At the same time the following six partitions of 10 are the only ones having the property that in each of them, every part of the partition is of the form $5m+1$ or $5m+4$ $(m = 0, 1, 2, \ \ .)$:

$$9+1,\ 6+4,\ 6+1+1+1+1,\ 4+4+1+1,\ 4+1+1+1+1+1+1,\ 1+1+1+1+1+1+1+1+1+1. \qquad (6.2)$$

Ramanujan identified the fact that the number of partitions satisfying each of the two properties is the same for every positive integer. Ramanujan's insight and brilliance in seeing through this property is remarkable and hence I call this result as a masterpiece or a gem. This result is now known as Rogers–Ramanujan identity as it was proved independently by both of them. It can be stated as:

$$[p(n)]_{d\geq 2} = [p(n)]_{5m+1 \text{ or } 5m+4}. \qquad (6.3)$$

–♦–

7 Consider a square PQRS with side = 4, and center at C (*Figure* 7.1). Divide it into 4 subsquares, each of side 2. Let the centers of these subsquares be L,M,N,G.

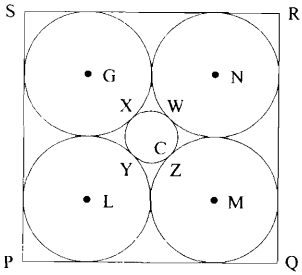


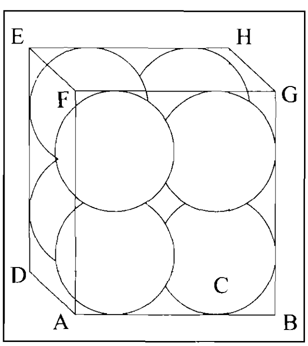
Figure 7.1.

In each of these subsquares, draw a circle, with radius = 1, which touches all the 4 sides of the subsquare. Now draw a circle with center at C, which touches all the four above mentioned circles at X,Y,Z,W. It is easy to see that CN=CM=CL=CG=$\sqrt{2}$. The radius of the innermost circle Γ = CW = $\sqrt{2}-1 < 2$. So Γ lies completely inside the original square PQRS.

Now consider the 3-D version of this example. It consists of a cube ABCDEFGH (see *Figure* 7.2). Divide it into 8 subcubes. Each of these 8 subcubes will be inscribed with a sphere with radius = 1 and which touches all the faces of the corresponding subcube. Now draw a sphere Γ with its centre at the centre of the original cube ABCDEFGH and touching all the above mentioned 8 spheres. It can be seen that the radius of $\Gamma = \sqrt{3}-1 < 2$. So this innermost sphere Γ will also lie completely inside the original cube ABCDEFGH.

Consider now the n-D version of this example. We shall have n-D hypercube D with side = 4. It will be divided into 2^n sub-hypercubes, each with side = 2, and a hypersphere with radius = 1 inscribed in each of them, which will touch all the hyperfaces of the corresponding subhypercube. Finally we shall have the innermost hypersphere with radius = $\sqrt{n}-1$.

Figure 7.2.



If $n = 9$, the innermost 9-D hypersphere's radius = $\sqrt{9}-1 = 2$; so it will touch all the faces of the original hypercube D. If $n > 9$, i.e. if we are working in n-D with $n > 9$, the innermost hypersphere's radius = $\sqrt{n}-1 > 2$; so when $n > 9$, the innermost hypersphere will pop outside the original hypercube D with side = 4, beating all our imagination and experience. I call this example a gem as it is counterintuitive and teaches us the important lesson that what seems obvious and intuitive need not be true [8].

–♦–

8. We saw a counterintuitive example in section 7 above. Certain results in mathematics follow intuitively. On the other hand, look at the dichotomy of odd primes into those of the type $4k + 1$, and those of the form $4k + 3$. An odd prime of the form $4k + 1$ can be expressed uniquely as a sum of two perfect squares, while a prime (in fact, any odd integer) of the type $4k + 2$ cannot at all be represented as a sum of two perfect squares. This dichotomy is non-intuitive to me, and so I was wondering whether a result exists about the possibility of expressing any positive integer n as a sum of two perfect squares. And I found that such a result indeed exists. It states that a positive integer n can be expressed as a sum of two perfect squares in $w(n)$ ways, where

$$w(n) = 4[d_1(n) - d_3(n)], \tag{8.1}$$

where $d_1(n)$ = number of divisors of n that are of the type $4k + 1$, $d_3(n)$ = number of divisors of n that are of the type $4k + 3$.

Here the expression a^2+b^2 is to be counted different than $b^2 + a^2$. Similarly, $(-c)^2 + d^2$ is to be counted different from $c^2 + d^2$, and the like.

As an example, take $n = 25$. Its divisors are 1, 5, and 25. So $d_1(25) = 3, d_3(25) = 0$, nd therefore $w(25) = 12$. These 12 ways are: $(\pm3)^2+(\pm4)^2$, $(\pm4)^2+(\pm3)^2$, $(\pm5)^2+0^2$, $0^2 + (\pm5)^2$. (Note: In mathematics, zero has no sign, though in computer arithmetic, distinction is made between +0 and −0 sometimes).

The result (8.1) is so beautiful, pleasing, and satisfying that I consider it a mathematical gem.

–♦–

9. Now we consider a gem which enlightens us about the mathematical concept of a number. Generally, we write a number as a sequence of digits from left to right.

Suggested Reading

[1] E T Bell, *Men of Mathematics*, Touchstone, 15 October 1986.

[2] David Wells, *You are a Mathematician*, Chapter IV (The Games of Mathematics) Problem 4E (Masterpiece by Euler), Gardners Books, 26 October 1995.

[3] Burkard Poleter, *Beauty in Mathematical Proof*, Walker & Co., 2004.

[4] J N Kapur, Fascinating Prime Numbers, *Indian Journal of Mathematical Education*, Vol.11, No.3, pp.111–118, Oct. 1991.

[5] Ross Honsberger, *Mathematical Gems II*, Chapter 4 ('The Generation of Prime Numbers, The Mathematical Association of America), (The Dolciani Mathematical Expositions, No. 2), pp.29–37, 1976.

[6] Martin Gardner, Mathematical Games, *Scientific American*, March 1975.

[7] R Athmaraman (Ed), *The Wonder World of Kaprekar Numbers*, The Association of Mathematics Teachers of India, p.91, 2004.

[8] Asha Rani Singh, Mathematics: An unexpected Pleasure, *The Mathematics Student*, Vol.65, Nos.1–4, p.24, 1991.

[9] David Wells, *You are a Mathematician*, Chapter III (Mathematics as Science), Problem 3J, Gardners Books, 26 October 1995.

Figure 9.1.

$$
\begin{array}{r}
\underline{5} \\
\underline{2}5 \\
\underline{6}25 \\
39\underline{0}625 \\
152587\underline{9}0625 \\
\bullet \\
\bullet \\
\bullet
\end{array}
$$

Now follow the following procedure of constructing a sequence of digits: Take a digit, say 5. Square it. Continue to square the result of previous operation of squaring, infinitum. A partial list of the results of these operations is given in *Figure* 9.1. Using it, construct a sequence S of digits from right to left, choosing the nth ($n = 1, 2, 3,\ \ .$) last digit in the nth number, in *Figure* 9.1, as the nth digit, from the right, of S. (The digit chosen from each number in *Figure* 9.1 is underlined). So

$$S = \qquad 90625 \tag{9.1}$$

Thus the sequence S is well-defined. Is the sequence S, a number as per mathematical concept of a number? You can find that

$$\text{if } T = 1 - S, \text{ then } S^2 = S,\ T^2 = T \quad \text{and } S \times T = 0. \tag{9.2}$$

A genuine number does not display these properties. So we learn an important lesson that a sequence of digits strongly resembling usual number need not be genuine number [9]. It is an eye-opener. Hence I call this example, a gem.

I feel that each of the gems cited above enhances our understanding and appreciation of mathematics.

Address for Correspondence
V G Tikekar
c/o Thakar
New Mahadwar Road
Near Padmaraje School
Kolhapur 416006, India.