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Quantum States of Neutron in Earth's Gravitational Field: A Challenge from the 36th International Physics Olympiad

This article describes a challenging problem concerning a neutron under gravity that was posed as a theoretical problem in the 36th International Physics Olympiad held in Salamanca, Spain from July 3-12, 2005. The Olympiad marked India's eighth foray into this exciting event where seventy-five nations participated. Our performance was a success and all five student members of the team secured medals which included two gold, two silver and one bronze medals.

1. Preamble

The International Physics Olympiad (IPhO) is an annual event initiated by erstwhile East European nations three and a half decades ago. India has been a latecomer to this event which is part festival, part competition – in short a celebration of the best in pre-university physics. India participated for the first time in 1998. This year marked our eighth foray into this exciting event. One of us (VAS)was the theory coordinator responsible for the training of five pre-university (Standard XII) students who participated in the 36th International Physics Olympiad at Salamanca, Spain from July 3 12, 2005[1]. He participated in the selection and training of our team.

Ravi Bhattacharjee, SGBT Khalsa College, Delhi was the delegation leader and Rajesh B Khaparde from HBCSE, Mumbai was the pedagogical leader of the Indian team. Bhupati Chakrabarti, City College, Kolkata was the scientific observer.

The road to the Physics Olympiad is indicated in *Table* 1. Our intense involvement begins in May when we run the selection camp at the Homi Bhabha Centre for Science Education (HBCSE), Mumbai. It is perhaps fair to say that at least a dozen of us have to make do with four hours of sleep a day for the next two months. We had a difficult time selecting the five students from the fifty-five who were present at the camp. This done, we

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Neutron, gravitational field, Bohr-Sommerfeld-Wilson quantization, projectile motion, elastic collision, Olympiad.

No.	Exam	Duration	Date	Participants	Assoc.
1.	NSEP	2.5 hrs	Nov./Dec.	>30,000	IAPT
2.	INPhO (Theory)	4 hrs	Jan. end	250	IAPT + HBCSE
3.	OCSC	10 days	May-June	55	HBCSE
4.	IPhO-PDT	10 days	July	5	HBCSE
5.	IPhO	10 hrs	July	> 350	Intnl. location!

NSEP – National Standard Examination in Physics (An annual nationwide examination conducted by the voluntary body, IAPT). INPhO – The Indian National Physics Olympiad (Exam) (An annual national level examination conducted jointly by HBCSE and IAPT). OCSC – Orientation Cum Selection Camp (This is run for about ten days in May-June at HBCSE with the help of a national pool of physics Faculty). IAPT – The Indian Assoction of Physics teachers (A voluntary organition of physics teachers active for over 20 years). HBCSE – The Homi Bhabha Centre for Science Education (National Centre set up by TIFR in Mumbai, India). IPhO – The International Physics Olympiad (Exam) (An annual examination in which seventy-five nations participated at Salamanca, Spain in 2005).

Table 1. The Road to the Physics Olympiad.

conducted a pre-departure training (PDT) for the five students before flying them to Spain for the IPhO. More details of the history, procedure, and preparation can be found elsewhere [1,2].

The examination consists of two components: experiment and theory. Each component is of five hours duration. The theory component consists of three questions each of ten points. The experimental component carries a weightage of 20 points. As mentioned above, our efforts were met with a fair degree of success. We secured two gold, two silver and one bronze medals. In short, each one of our five participants secured a medal. One student (N Tejaswi Venumadhav) missed gold only by 0.2 points. A total of seventy-five nations participated in this event.

2. Quantum Effects of Gravity: The Problem

The problem posed in the 36th International Physics Olympiad happens to be a front line research problem. It was based on the work of V V Nesvizhevsky *et al.*

No.	Name	Place	Points	Medal
1.	Piyush Srivastava	Allahabad	47.4/50	**Gold**
2.	Sameer Madan	Panchkula	45.4/50	**Gold**
3.	N Tejaswai Venumadhav	Hyderabad	44.8/50	**Silver**
4.	M Hema Chandra Prakash	Hyderabad	41.6/50	**Silver**
5.	Arjun Radhakrishna	Bangalore	36.1/50	**Bronze**

Note that the gold medal cutoff was 45/50 and Mr Tejaswi Venumadhav missed it by a bare 0.2 points!

Table 2. Results of the 36th International Physics Olympiad.

and involved an international collaboration between France, Russia and Germany. Their work was published as an article in *Nature* (2002)[3] and *Physical Review* D, (2003) [4]. The fact that it was posed at the international physics olympiad and was successfully solved by a number of higher secondary school students is a reflection of the very high standard at the physics olympiad.

This problem involves the quantum effects of gravity. A neutron is projected to fall under gravity and it is confined using mirror and absorber arrangement (see *Figure* 1). An application of the Bohr-Sommerfeld-Wilson quantization rule [5,6] yields surprising results: (i) Only discrete energy levels are allowed for the neutron. (ii) Only discrete projection heights are possible. The latter effect is space-quantization in a literal sense [7].

The problem is presented in an abbreviated form to make it suitable for presentation as an article. We shall discuss the solution in a formal fashion without getting into distracting "numerics" The readers are however encouraged to look up the original papers [3,4] as well as investigate other problems of 36th International Physics Olympiad [8].

As mentioned earlier, a group of physicists reported the experiment in which neutrons moving horizontally were allowed to fall towards a horizontal surface (mirror), where they bounced back elastically up to the initial height repeatedly until they encountered a detector.

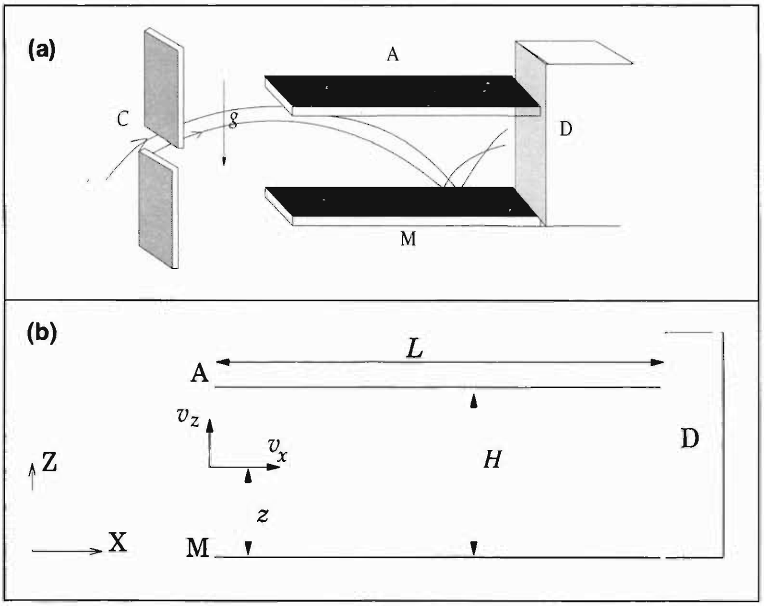

Figure 1. Schematic diagrams of the experiment: (a) The arrangement. (b) Illustration of the parameters employed in the text.

The setup of the experiment is sketched in *Figure* 1. It consists of the opening C, the neutron mirror M (at height $z = 0$), the neutron absorber A (at height $z = H$ and with length L) and the neutron detector D. The beam of neutrons flies with constant horizontal velocity component v_x from C to D through the cavity between A and M. All the neutrons that reach the surface of A are absorbed and disappear from the experiment. Those that reach the surface of M are reflected elastically. The detector D counts the transmission rate $N(H)$, that is the total number of neutrons that reach D per unit time.

The neutrons enter the cavity with a wide range of positive and negative vertical velocities, v_z. Once in the cavity, they are confined between the mirror below and the absorber above. Next we enumerate the problem in steps.

1. Compute classically the range of vertical velocities $v_z(z)$ of those neutrons which on entering at a height z, can arrive at the detector D. Assume that L is sufficiently large.

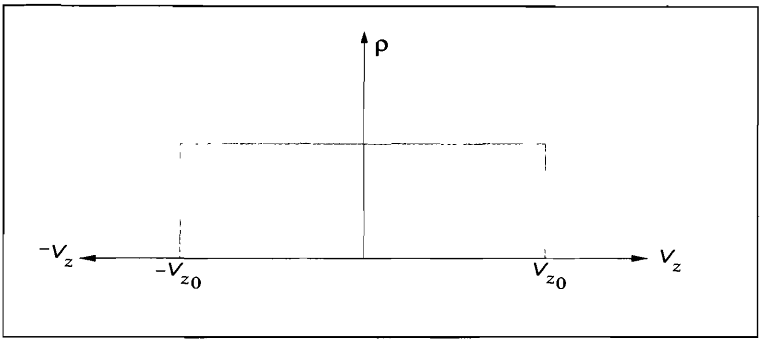

Figure 2. Plot of probability density of neutrons per unit time, per unit vertical velocity and per unit height. Here

$v_{z0} = \sqrt{2g(H-z)}$

2. Calculate classically the minimum length L_c of the cavity which ensures that all neutrons outside the previously computed velocity range are absorbed. Use $v_x = 10.0$ ms^{-1} and $H = 50.0\,\mu$m.

3. Compute the classical neutron transmission rate $N_c(H)$ assuming that neutrons arrive at the cavity with vertical velocity v_z and at height z. We assume that all values of v_z and z are equally probable. The constant number of neutrons (per unit time, per unit velocity, per unit height) that enter the cavity with vertical velocity v_z and at height z is represented by ρ (see *Figure* 2). $N(H)$ is measured at D. Classically we expect that it increases monotonically with H

4. The experimental results obtained by that group disagree with the above classical predictions, showing instead that the value of $N(H)$ experiences sharp increases when H crosses some critical heights H_1, H_2 etc. (see *Figure* 3). In other words, the experiment showed that the vertical motion of neutrons bouncing on the mirror is quantized.

   The Bohr–Sommerfeld–Wilson quantization rule [5,6] states that

$$\oint p_z dz = nh, \quad (n = 1, 2, 3 \quad .)$$

where h is Planck's constant, p_z is the vertical

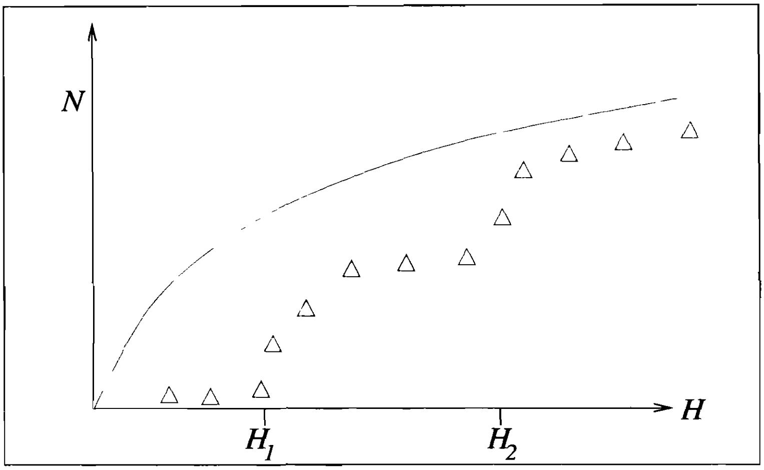

Figure 3. Plot N(H) of neutrons, observed experimentally by Grenoble group showing that the value of N(H) experiences a sharp increase when H crosses some critical heights H_1, H_2, etc. Solid smooth curve corresponds to the classical dependence N_c (H) (see equation (4) on p.98).

component of the momentum and the symbol $\oint$ covers one whole bouncing cycle. Only neutrons with these quantized values are allowed in the cavity.

Obtain the expression for discrete heights H_n and energy levels E_n, associated with the vertical motion, using the Bohr–Sommerfeld–Wilson quantization condition. Give the numerical result for H_1 in μm and for E_1 in eV

5. The uniform initial distribution ρ of neutrons at the entrance changes during the flight through a long cavity into the step like distribution detected at D (see *Figure* 3). We consider for simplicity the case of a long cavity with $H < H_2$. Classically, all neutrons with energies in the earlier considered range were allowed through it, while quantum mechanically only electrons of energy E_1 are permitted. According to the Heisenberg uncertainty principle, this reshuffling requires a minimum time of flight.

   Estimate the minimum time of flight t_q and the minimum length L_q of the cavity needed to observe the first sharp increase in the number of neutrons at D. Use $v_x = 10.0\ \text{ms}^{-1}$.

6. In quantum theory, the number of neutrons per unit time per unit height for a long enough cavity and $H > H_1$, is given by

$$I(z) = \begin{cases} TE_1\rho/p_z(z) & \text{if } z < H_1 \\ 0 & \text{otherwise} \end{cases}$$

where T is a dimensionless constant.

Compute the quantum rate $N_q(H)$ for cavity heights H between 0 and H_2 and compare with $N_c(H_1)$.

3. Solution

1. We assume that L is much larger than any other length in the problem.

Since the energy of the particle is conserved, so

$$\frac{1}{2}mv_x^2 + \frac{1}{2}mv_z^2 + mgz = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_Z^2 + mgZ,$$

where z is the height at which the neutron enters the cavity and Z is the displacement along z-axis during its motion under gravity. Note m is the mass of one neutron and g is the gravitational acceleration.

Maximum height which a neutron can reach is H and at $Z = H$ vertical speed must be zero. Only those neutrons will reach the detector D, which cannot cross this height. Thus, conservation of energy implies

$$\frac{1}{2}mv_x^2 + \frac{1}{2}mv_z^2 + mgz \leq \frac{1}{2}mv_x^2 + 0 + mgH,$$

$$\text{i.e.,} \quad v_z^2 \leq 2g(H - z).$$

So the allowed values of v_z are in the range,

$$-\sqrt{2g(H - z)} \leq v_z \leq \sqrt{2g(H - z)}. \qquad (1)$$

Alternatively we can also obtain the above result by kinematics.

2. To obtain the critical length we assume the worst case scenario. Let the neutron be projected from $z = H$ and with downward velocity of $\sqrt{2gH}$ It reaches the detector after the first full bounce and its transit time is

$$t = 2\sqrt{\frac{2H}{g}}.$$

Then the critical length is

$$\begin{aligned} L_c &= v_x t \\ &= 2v_x\sqrt{\frac{2H}{g}}. \end{aligned} \qquad (2)$$

Substituting the given values, we obtain

$$L_c = 6.39 \text{ cm}. \qquad (3)$$

Similarly, for upward velocities $v_z > \sqrt{2g(H-z)}$, this maximum transit time will be almost half of the time obtained for downward velocities. So in this case, after calculation we can obtain $L_c =$ 3.19 cm. So, the minimum length L_c of the cavity to ensure that all neutrons outside the velocity range given by (1) are absorbed is 6.39 cm.

3. For the allowed v_z values (see equation(1)), classical rate of neutrons entering the cavity is equal to the number of neutrons detected at D. If we assume a uniform probability for all allowed v_z values (see *Figure* 2), then for neutrons entering the cavity between heights z and $z + dz$, the number $dN_c(H)$ reaching the detector per unit time is

$$dN_c(H) = \rho \times 2\sqrt{2g(H-z)} \times dz.$$

Hence the total number of neutrons is obtained by integrating over all possible heights

$$\begin{aligned} N_c(H) &= \int_0^H 2\rho\sqrt{2g(H-z)}\,dz, \\ N_c(H) &= \frac{4}{3}\rho H\sqrt{2gH} \end{aligned} \qquad (4)$$

Solid smooth curve of *Figure* 3 depicts the plot of $N_c(H)$ versus H

4. As suggested, the Bohr–Sommerfeld–Wilson quantization condition states that

$$\oint p_z dz = nh, \qquad (n = 1, 2, 3 . \; .),$$

where the symbol $\oint$ covers one whole bouncing cycle. Thus for a neutron bouncing off a perfect reflector, we obtain its quantized states, by employing this Bohr–Sommerfeld–Wilson quantization condition

$$2 \int_o^{H_n} m\sqrt{2g(H_n - z)} dz = nh.$$

After carrying out integration,

$$H_n = \left(\frac{3h}{4m\sqrt{2g}} \right)^{2/3} \times n^{2/3} \qquad (5)$$

And energy is given by,

$$E_n = mgH_n = mg \left(\frac{3h}{4m\sqrt{2g}} \right)^{2/3} \times n^{2/3} \qquad (6)$$

Substituting the values for h, g and the neutron mass $m = 1.67 \times 10^{-27}$ kg, for $n = 1$ we obtain,

$$H_1 = 16.50\,\mu\text{m}, \qquad (7)$$
$$E_1 = 1.69 \times 10^{-12}\ \text{eV}. \qquad (8)$$

This energy corresponds to a temperature $\approx 2.0 \times 10^{-8}$ K. Thus we need a source of ultra-cold neutrons. Here we can note some interesting points:

(a) The height H_1 is of the same order as the cavity height $H = 50.0\ \mu$m. This opens up the possibility of observing spatial quantization.

(b) The dependence of energy on n is fractional ($\propto n^{2/3}$). This is distinct from the hydrogen atom ($\propto -1/n^2$), the harmonic oscillator ($\propto n$) and the particle in a box ($\propto n^2$). Note that the exact solution based on the Schrodinger equation yields $E_n \propto \left(n - \frac{1}{4}\right)^{2/3}$ [4].

(c) The transition from $n = 2$ to $n = 1$ will yield a photon of wavelength 1.24×10^6 m which belongs to ULF (Ultra Low Frequency) radio range.

5. Using Heisenberg Uncertainty Principle,

$$\begin{aligned} \triangle E \, \triangle t &\simeq \frac{\hbar}{2}, \\ \triangle t &\simeq \frac{\hbar}{2 \triangle E}. \end{aligned}$$

From equation (8), $\triangle E \simeq 10^{-12}$ eV So $\triangle t \simeq 0.5 \times 10^{-3}$ s.
Therefore estimated value of t_q is 0.5 ms and $L_q = v_x t_q = 0.5$ cm.

Thus the cavity length needs to be at least half a centimeter for the quantum jump in the neutrons to be observed.

6. For $H < H_1$, $N_q = 0$.
For $H_2 > H > H_1$,

$$\begin{aligned} dN_q &= I(z)\, dz. \\ N_q &= \int_0^H I(z)\, dz \\ &= TE_1\rho \int_0^{H_1} \frac{1}{p_z} dz \\ &= \frac{2TE_1\rho}{m}\sqrt{\frac{H_1}{2g}}. \end{aligned}$$

$$N_q(H) = \begin{cases} 0 & H < H_1 \\ \frac{2TE_1\rho}{m}\sqrt{\frac{H_1}{2g}} & H_1 < H < H_2 \end{cases}$$

$$\frac{N_q(H > H_1)}{N_c(H_1)} = \frac{2TE_1\rho\sqrt{H_1/2g}}{m\,\frac{4}{3}\,\rho H_1\sqrt{2gH_1}} = \frac{3TE_1}{2\,m\,2\,gH_1}.$$

Keeping in mind that $E_1 = MgH_1$

$$N_q(H > H_1) = \frac{3T}{4}N_c(H_1).$$

Figure 3 depicts the step like behavior of the detected neutrons. As mentioned in part(4) of the solution, the energy of the neutrons corresponds to a temperature of 2×10^{-8} K. The Grenoble facility in France is equipped with such a source of ultra-cold neutrons. The de-Broglie wavelength of these neutrons $\lambda_{dB} \approx 22\,\mu$m. This is comparable to the cavity height of $50\,\mu$m.

Long standing efforts to measure the quantum effects in the presence of gravity were frustrated by a number of factors. (i) Gravitational effects are by far overshadowed by electromagnetic effects. The choice of neutrons was dictated by this consideration. Since the neutron has a magnetic moment, magnetic shielding was provided for the setup. (ii) On earth, the effect of gravity is unidirectional. The mirror arrangement ensured confinement and bound states. What is remarkable is that the quantum effect in the presence of gravity, although expected, has actually been observed. What is also remarkable is that one of our students (Piyush Srivastava) solved the entire problem correctly (10/10) and two other students Sameer Madan (9.4/10) and N Tejaswi Venumadhav (9.2/10) secured more then 90% marks in this question.

Acknowledgment

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Suggested Reading

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