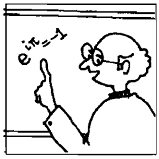

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. “Classroom” is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Utpal Mukhopadhyay
Barasat Satyabharati Vidyapith
P.O. Nabapally 700126
Dist. North 24- Parganas
West Bengal, India

Some Interesting Features of Hyperbolic Functions

Introduction

Hyperbolic and trigonometric or circular functions have similarities as well as differences. Both can be expressed in terms of exponential functions, both can be represented by infinite series, both are periodic functions, and so on. On the other hand, representations of the hyperbolic functions in terms of the exponential function do not contain the imaginary unit i, e.g., $\cosh x = (e^x + e^{-x})/2$, whereas for the trigonometric functions, they do, e.g., $\cos x = (e^{ix} + e^{-ix})/2$. Also, the periods of hyperbolic functions are non-real quantities, whereas those of trigonometric functions are real. In this article, we describe some features of hyperbolic functions.

Simple Properties

The following results relating to hyperbolic functions will be used.

Keywords
Hyperbolic functions, catenary, Riccati equation.

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \tag{1}$$
$$\sinh 2x = 2\cosh x \sinh x, \tag{2}$$

Box 1.

Abraham De Moivre (1667-1754) was born in the province of Champagne in France. He mastered mathematics on his own, and his mathematical talent brought him close to Newton. In 1697, he was elected a member of the Royal Society. In 1722, he proposed the famous theorem which bears his name, but never published it. He was one of the members of the commission set up by Royal Society for setlling the well-known priority dispute between Newton and Leibniz.

$$\begin{aligned}\sinh(x+y) &= \sinh x\cosh y + \cosh x\sinh y, &(3)\\ \cosh(x+y) &= \cosh x\cosh y + \sinh x\sinh y, &(4)\\ \sinh(-x) &= -\sinh x, &(5)\\ \cosh(-x) &= \cosh x. &(6)\end{aligned}$$

Hyperbolic Analogue of de Moivre's Theorem

From the identity $\cosh x + \sinh x = e^x$, which follows directly from the definitions of cosh and sinh and holds for all x, we deduce that if $n \in Z$, then the value of $(\cosh x + \sinh x)^n$ is $\cosh nx + \sinh nx$, and if $n \in Q$, then one value of $(\cosh x + \sinh x)^n$ is $\cosh nx + \sinh nx$.

Significance of the Parameter θ for Hyperbolic Functions

Consider the unit circle $x^2+y^2 = 1$ with centre O (*Figure* 1). Drawing upon the identity $\cos^2\theta + \sin^2\theta = 1$, we write its equation in parametric form as $x = \cos\theta$, $y = \sin\theta$. The parameter θ has a clear geometric significance here: if $\mathrm{P}(\cos\theta, \sin\theta)$ is an arbitrary point on the circle, then θ is the angle between OP and the positive x-axis. For this reason, trigonometric functions are known as the *circular functions.*

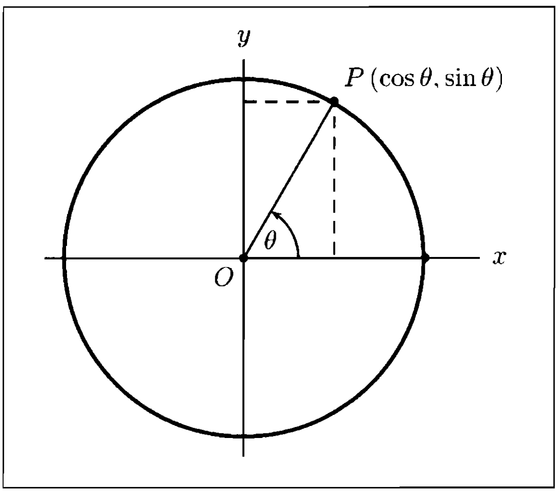

Figure 1.
Graph of $x^2 + y^2 = 1$.

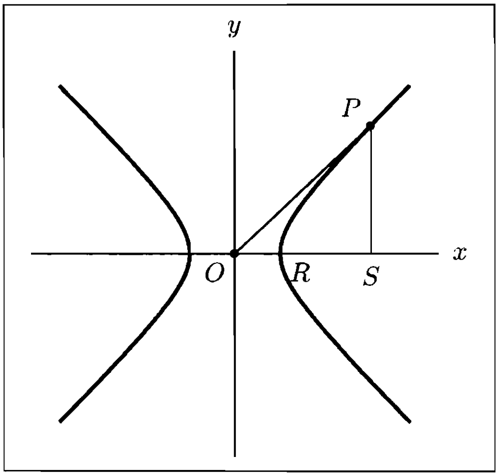

Figure 2. Graph of $x^2-y^2=1$.

What corresponding statement can be made for the hyperbolic functions? The answer is much less obvious. Consider any point P (x, y) on the rectangular hyperbola $x^2 - y^2 = 1$ (*Figure* 2). Let S $(x, 0)$ be the foot of the perpendicular from P to the x-axis, and let R $(1, 0)$ be the vertex of the hyperbola. Let O $(0, 0)$ be the origin. Invoking the identity $\cosh^2\theta - \sinh^2\theta = 1$, we write the equation of the curve in parametric form as $x = \cosh\theta$ and $y = \sinh\theta$. We must now find some meaning to the parameter θ.

The area of region OPR is given by Area (OPR) = Area ($\triangle$OPS) – Area (PRS). Using integration, we get:

$$\begin{aligned}
\text{Area (OPR)} &= \frac{1}{2}xy - \int_1^x \sqrt{u^2-1}\,du \\
&= \frac{1}{2}\cosh\theta\sinh\theta - \int_0^\theta \sinh^2 t\,dt \\
&= \frac{1}{2}\cosh\theta\sinh\theta - \int_0^\theta \frac{1}{2}(\cosh 2t - 1)\,dt \\
&= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\left(\frac{\sinh 2\theta}{2} - \theta\right) \\
&= \frac{\theta}{2}.
\end{aligned}$$

Therefore, $\theta = 2 \times \text{Area (OPR)}$.

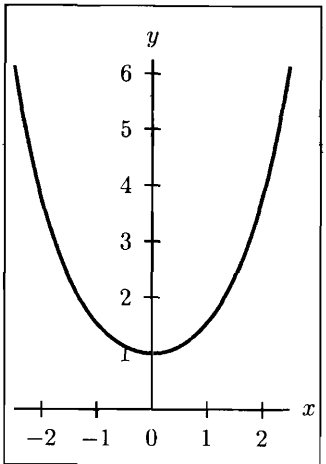

Figure 3. Graph of y = cosh x.

Box 2.

Vincenzo Riccati (1707-1775) was an Italian Jesuit. Struck by the smilarity of the equations $x^2 + y^2 = 1$ and $x^2 - y^2 = 1$, he went on to develop his theory on hyperbolic functions from a geometrical standpoint. The famous Riccati differential equation,

$$\frac{dy}{dx} = p(x)y^2 + q(x)y + r(x)$$

comes from his father, Jacopo Riccati (1676-1754). The Riccati family is another famous family of mathematicians in Europe like the Bernoullis, though not as prolific.

So for the hyperbola $x^2 - y^2 = 1$, if we parametrize the curve as $x = \cosh\theta$, $y = \sinh\theta$, then the parameter θ represents twice the area of the region enclosed by the curve, the x-axis and the line segment from the origin to the point $(\cosh\theta, \sinh\theta)$. This explains the reason behind the name *hyperbolic functions.*

The Catenary and Hyperbolic Functions

If a flexible string is suspended under gravity by its two ends, then the string hangs in the form of a catenary (*Figure* 3). The general equation of a catenary is

$$y = a\cosh\frac{x}{a} + b, \tag{7}$$

where a, b are constants; a depends on the mass per unit length and tension of the string, and b depends on the position of the x-axis. As a increases, the catenary becomes narrower and deeper. If we take b to be zero, then the equation reduces to

$$y = a\cosh\frac{x}{a}. \tag{8}$$

For $a = 1$, the equation becomes $y = \cosh x$.

In 1757, Vincenzo Riccati for the first time used the notations Ch x and Sh x for the functions $(e^x + e^{-x})/2$ and $(e^x - e^{-x})/2$ respectively. These have got modified over time into their present forms.

There is a striking similarity between the shape of a catenary and that of a parabola. The similarity can be explained by using the power series for $\cosh x$; since

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \end{aligned}$$

by addition, we get:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} +$$

For $x \approx 0$, the powers of x beyond x^2 become insignificant, so:

$$\cosh x \approx 1 + \frac{x^2}{2!}. \tag{9}$$

The right side represents a parabola, so we conclude that for small x, a catenary can be approximated by a parabola. It is worth mentioning here that of all the possible curves in which a suspended string can hang, the catenary has the least potential energy.

Final Remarks

The various formulae related to hyperbolic functions can all be proved using the definition of hyperbolic functions in terms of the exponential function. In this connection, I would like to ask whether it is possible to prove the addition formulae for cosh and sinh using geometrical methods, as can be done for the trigonometric functions. Till now, I have failed to do so. It is possible that geometrical methods are unsuitable for proving those formulae. Any suggestions or constructive comments in this line would be most welcome.

A possible geometrical interpretation of the addition formulae is the following. Consider the points A $(x, y) = (\cosh a, \sinh a)$ and B $(x', y') = (\cosh b, \sinh b)$ on the hyperbola $x^2 - y^2 = 1$, corresponding to the parameter values a and b. The addition formulae give the point C $(xx' + yy', xy' + yx')$, corresponding to the parameter value $a + b$. The slopes of OA, OB and OC are now seen to be related through the equation

$$\text{slope (OC)} = \frac{\text{slope (OA)} + \text{slope (OB)}}{1 + \text{slope (OA)} \ \ \text{slope (OB)}}. \tag{10}$$

It would be of interest to find an independent proof of this relation.

Suggested Reading

[1] Eli Maor, *The Story of a Number*, Universities Press, 1999.

[2] Eli Maor, *Trigonometric Delights*, Universities Press, 2000.

[3] Shailesh Shirali, *A Primer on Logarithms*, Universities Press, 2002.

[4] BC Das and B N Mukherjee, *Higher Trigonometry*, UN Dhur & Sons Pvt. Ltd., 1996.

I would like to ask whether it is possible to prove the addition formulae for cosh and sinh using geometrical methods, as can be done for the trigonometric functions. Any suggestions or constructive comments in this line would be most welcome.