Solution of the Cubic

A Simple Version of Cardano's Formula

Jasbir S Chahal

1. Introduction

Cardano's formula for solving a cubic is the crowning achievement of renaissance mathematics. Yet, it does not receive the same recognition in our curricula as does the quadratic formula, which was discovered long before it. It is rather surprising that there have not been attempts to simplify further the messy formulas of Cardano (see ([1],pp.606–616), or ([2], pp.187–189)) to a form that would be easier for the students to remember. Apparently the messy nature of the formulas for solving the cubic is a reason for the lack of their popularity. Another reason could be the Galois theory, which modern authors use in their exposition of Cardano's formula. We show that a simple trick, namely a rescaling of the discriminant, reduces not only the formula to a simpler form, but also its verification to a trivial calculation, with no reference to Galois theory. Although Galois theory is an indispensable tool in algebra and number theory, it is not necessary to wait until one learns it, for Cardano's formula. Cardano's formula can be introduced in a first course on complex numbers.

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Part 1. Foundation of Basic Arithmetic, *Resonance*, Vol.11, No.1, January 2006.
Part 2. Foundation of Basic Arithmetic, *Resonance*, Vol.11, No.2, February 2006.
Part 3. Foundations of Basic Arithmetic, *Resonance*, Vol.11, No.7, July 2006.

By the celebrated theorem of Abel–Ruffini, a general equation of degree five or more is not solvable by radicals, whereas solving a quartic equation can be reduced to solving a cubic equation. Thus Cardano's formula filled the essential gap in our understanding of the solutions of polynomial equations. The purpose of this article is to present a lowbrow exposition of Cardano's formula than that found in the literature and to tell the story behind its discovery in order to put the matter in a proper historical perspective.

Keywords
Cubic, discriminant, roots of unity, radicals.

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2. History

There are reasons to believe that the Babylonians of 2000 BC were familiar with solving quadratic equations, albeit neglecting the negative solutions. Much later, Brahmagupta (c.628 AD) and then al Khwarizmi (780–850 AD) described the quadratic formula more or less as we know it today.

The next step was to solve the cubic. The Arabs and the Chinese worked out special cases of the cubic numerically. But it took almost a thousand years from Brahmagupta's time to find a general solution to the cubic, *often attributed to the Italian Cardano. The story of* its discovery is as dramatic as it can be in the world of mathematics.

To solve any polynomial equation it suffices to take the leading coefficient equal to 1. Moreover, the so called *Viète substitution* $X = x - A/3$ reduces the cubic equation

$$X^3 + AX^2 + BX + C = 0$$

to one of the form $x^3 + ax + b = 0$.

Thus there is no loss of generality in assuming that the general cubic equation has no square term.

Hindu, Islamic or even the Italian algebra of Cardano's time was entirely rhetorical. There were no symbols for an unknown or its powers. Everything was communicated in words, and to facilitate memorization, formulas were stated as verses. For example, here is part of a verse (see [3],p.36) for the equation $x^3 + px = q$:

Squeaxno, adtwix
Noesquax, adsub
Axesquno, subadsub

It was only after Cardano had published the solution of the cubic in 1545 that Francois Viète (1540–1603) introduced, in his book *The Analytic Art*, our present usage of letters to represent unknown quantities. He used vowels for variables and consonants for constants. However, we owe our tradition of using earlier letters $a, b, c,$ for constants and the later ones $x, y, z,$ for variables to Descartes. Viète had no symbol for equality. It was Robert Recorde who introduced the symbol $=$ for equality in 1557. The signs $+$ and $-$ appeared for the first time in Germany at the end of the fifteenth century as symbols for surplus and deficit in business records. In 1514, the Dutch mathematician Vander Hoecke became the first to use them in algebraic expressions. Thomas Harriot was the first (in 1631) to use a dot for multiplication, and he is also responsible for the inequality signs $<$ and $>$. In the same year (1631) William Oughtred introduced the cross sign $\times$ for multiplication. The square root symbol $\sqrt{\ }$ was invented by Christoff Rudolff (1510–1558), though some historians dispute it. In 1655, John Wallis was the first to use the symbol ∞ for infinity, probably suggested by the late Roman symbol ∞ for a millennium. For more, see [4].

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Leibniz used these symbols in his calculus which was popularized by the Bernoullis. The Bernoulli family had great influence on Euler. Finally, it was Euler who utilized these symbols throughout his writings and made them the language of mathematics. Thus the mathematical symbols, which look very intimidating to many people, are very recent phenomena. But they facilitated great advances in mathematics.

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Arabs, and the early Europeans who were to take off from where the Arabs had left, did not consider negative coefficients. Thus there were dozens of cases of the cubic equation to be considered. For example, the so-called *depressed form* alone, with square term absent, was split

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into three cases:

$$x^3 + px + q = 0, \quad x^3 = px + q \text{ and } x^3 + px = q \quad (1)$$

with $p, q > 0$.

It was Scipione del Ferro (1465–1526), a professor at the University of Bologna, who was the first to find a method of solving equations (1) somewhere around 1510. To insure priority, modern professors announce results even before they have fully checked their proofs. But the academic life in sixteenth century Italy was quite different. There was no tenure. University appointments were mostly temporary, subject to periodic renewal. The most common way for a professor to stay in his position was to win public contests. A new contender for his job would exchange with him a list of problems to be solved by the other in a specified amount of time. It was required that a solution to every problem submitted must exist. Sometime later they would meet each other in a public forum to present their solutions; so it was a good strategy for professors to keep their discoveries secret and use them for these public contests. Professor del Ferro never had the occasion to use his solution for such a contest and just before his death in 1526, secretly passed it on to his student Antonio Fiore, as well as his successor Professor della Nave (1500–1558) at the University of Bologna. Even though they never publicized the solution, the news that someone had found a solution to the cubic started to circulate among Italian mathematicians. Another Italian Nicolo Tartaglia (1500–1557) from Brescia boasted to have the solution. This was too much for Fiore to take, so he challenged Tartaglia to a public contest. All of Fiore's word problems required the knowledge of a solution to the cubic equation. Having no solution of the cubic yet, Tartaglia was thus trapped, but during the time set aside he worked day and night and just before the contest on the night of 12 February 1535, found the solution to the cubic. Having worked out the solution himself, Tartaglia easily defeated Fiore who had inherited the solution from his teacher.

At that time Girolamo Cardano (1501–1576) was lecturing in Milan on algebra. When he heard about Tartaglia's solution he wrote to Tartaglia. He wanted to see the solution so that it could be included in his lectures on algebra. Tartaglia showed the solution only after extracting an oath from Cardano that it would not be included in Cardano's forthcoming book, even with full credit to him. Tartaglia wanted to publish it himself. Cardano kept his promise but assisted by his brilliant student Lodovico Ferrari (1522–1565), started working on the problem himself. Ferrari even managed to solve the fourth degree equation. But their solutions depended on reducing the problem to the cases solved by Tartaglia.

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Tartaglia still had not published anything. Cardano did not want to break his promise to Tartaglia, but felt a need to make the solution available to the public. Meanwhile, after hearing the rumor of the original solution by della Ferro, Cardano and Ferrari visited Professor della Nave in Bologna who graciously let them verify that del Ferro indeed had the solution. Cardano no longer felt an obligation to Tartaglia as he would only be publishing the same solution found independently some 25 years earlier by a mathematician now deceased. Thus in 1545, Cardano published his most important work, *Ars Magna* mainly devoted to the solution of the cubic. When the book appeared, Tartaglia was furious, even though Cardano had mentioned him as one of the original discoverers of the solution. To recoup his prestige, Tartaglia challenged Ferrari to a public contest, but this time he was defeated. To this day the method described in *Ars Magna* of solving the cubic equation is called Cardano's Formula. We now explain it from a modern point of view, which unifies all the cases into a single formula. For the original case-by-case discussion, see [3].

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3. Roots of a Complex Number

When we say a polynomial equation is solvable by radicals, we mean the solutions can be found in terms of expressions involving the four algebraic operations on the coefficients of the polynomial, and extracting their square roots, cube roots, and so on. So let $m > 1$ be an integer. We now indicate how to extract all the mth roots of a non-zero complex number.

Any complex number $z = x + iy \neq 0$ can be represented geometrically as

$$z = re^{i\theta} := r(\cos\theta + i\sin\theta),$$

where its *modulus* $r = \sqrt{x^2 + y^2} > 0$ and the *argument* $\mathrm{Arg}(z)$ of z is the angle $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Let $\sqrt[m]{r}$ be the positive real mth root of r and ω be the mth root of unity given by

$$\omega = \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}.$$

If we put $\alpha = \sqrt[m]{r}\, e^{i\theta/m}$, then the mth roots of z are α, $\omega\alpha$, $\omega^2\alpha$, $\ldots, \omega^{m-1}\alpha$ (which are clearly distinct, hence account for all of them). In particular, the ratio of any two of them is an mth root of unity. For example, the three cube roots of $8i$ are

$$2\left[\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}d\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}d\right)\right]$$

for $d = 0, 1, 2$.

4. Cardano's Formula

Recall that the quadratic equation

$$x^2 + bx + c = 0 \tag{2}$$

has two solutions

$$x = \frac{-b \pm \sqrt{\Delta}}{2},$$

where the quantity $\Delta = b^2 - 4c$ is called the *discriminant.* The discriminant *discriminates* the solutions. When there is no discriminant, that is, when $\Delta = 0$, the two roots are equal. In fact, the two roots are equal if and only if $\Delta = 0$.

Exercise. *Solve the quadratic equation (with complex coefficients)*

$$\left(1 - \frac{\sqrt{3}}{4} i\right) z^2 - \sqrt{5}\, z + 1 = 0.$$

To solve the cubic we may assume, as has been said earlier, that a general cubic equation is of the form

$$x^3 + ax + b = 0. \tag{3}$$

Again we can define its *discriminant* D such that no two solutions are equal if and only if $D \neq 0$. To aid memory, we make the formulas for the solutions of (3) resemble as much as possible that of $x^2 + bx + c = 0$, which are

$$x_1 = \frac{-b + \sqrt{\Delta}}{2} \text{ and } x_2 = \frac{-b - \sqrt{\Delta}}{2}.$$

For this, we modify the traditional definition $D = -(4a^3 + 27b^2)$ of the discriminant of $x^3 + ax + b$ slightly. Our definition of the *discriminant* Δ of $x^3 + ax + b$ is

$$\Delta = \frac{4a^3 + 27b^2}{27}. \tag{4}$$

Let ω_1 and ω_2 be the two imaginary cube roots of unity:

$$\omega_j = \cos \frac{2\pi j}{3} + i \sin \frac{2\pi j}{3} \quad (j = 1, 2).$$

It is easy to check that for $\omega = \omega_1$ or $\omega_2, 1 + \omega + \omega^2 = 0$. Moreover,

$$\omega_1^2 = \omega_2 \text{ and } \omega_2^2 = \omega_1. \tag{5}$$

Now using (4), it is easy to check that

$$\frac{-b+\sqrt{\Delta}}{2}\ \frac{-b-\sqrt{\Delta}}{2} = -a^3/27.$$

Choose cube roots

$$\alpha_1 = \sqrt[3]{\frac{-b+\sqrt{\Delta}}{2}} \text{ and } \alpha_2 = \sqrt[3]{\frac{-b-\sqrt{\Delta}}{2}} \qquad (6)$$

such that

$$\alpha_1 \ \ \alpha_2 = -\frac{a}{3}. \qquad (7)$$

Theorem. (del Ferro–Tartaglia–Cardano). *The three solutions of $x^3 + ax + b = 0$ are*

$$\alpha_1 + \alpha_2,\ \omega_1\alpha_1 + \omega_2\alpha_2,\ \omega_2\alpha_1 + \omega_1\alpha_2. \qquad (8)$$

The equations (6), (7) and (8) taken together are called Cardano's formulas.

Proof. Plug each number from (8) in (3) and use (5), (6) and (7).

Example. To illustrate Cardano's formula we take one of the simplest examples, namely

$$x^3 - 1 = 0.$$

Here $a = 0$ and $b = -1$, so $D = 1$. From (6), we get $\alpha_1 = 1$ and $\alpha_2 = 0$. It follows from (8) that the three roots of $x^3 - 1$ are $1 + 0, \omega_1 \ \ 1 + \omega_2 \ \ 0, \omega_2 \ \ 1 + \omega_1 \ \ 0$, that is, they are $1, \omega_1, \omega_2$. This agrees with what we already know, that is $1, \omega, \omega^2$ are the three solutions of $x^3 - 1 = (x-1)(x^2 + x + 1) = 0$, where $\omega = \omega_1$ or ω_2 is a primitive cube root of unity.

Exercise. Solve the following cubic equations:

i) $x^3 - 2x + 4 = 0$, ii) $x^3 + x^2 - 2x - 1 = 0$.

Remark 1. Our choice of the cube roots α_1 and α_2 is dictated by the proof. However, to find the three solutions of (3), it is obvious that any choice of α_1 and α_2 will suffice, because the other choices just permute the three numbers in (8).

Remark 2. If the reader is familiar with the group structure on the points of the elliptic curve

$$y^2 = x^3 + ax + b \tag{9}$$

with coordinates in $\mathbb{C}$, together with the point O at infinity, the solutions of (3) are the x-coordinates of points of order two on (9).

Suggested Reading

[1] D S Dumit and R M Foote, *Abstract Algebra*, John Wiley, 2004.
[2] B L van der Waerden, *Algebra*, Vol. I, Frederick Unger, 1970.
[3] Girolamo Cardano, *Ars Magna*, Dover, 1993.
[4] F Cajori, *A History of Mathematical Notations*, Dover, 1993.
[5] D J Struik, *A Concise History of Mathematics*, Dover, 1948.

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Errata

Resonance, Vol.11, No 7, July 2006

Page 6: second paragraph, 4th line
... crescent shaped infected blood cells should read,
... crescent shaped parasites inside the infected blood cells...

Page 42: The first two sentences should read,
The intensity of solar radiation in the Earth's direction from the Sun is approximately 1.353 kw/m^2, a number also called the 'Solar Constant'. Accordingly, it is estimated that Earth receives about 96 billion kw from the Sun constantly.'

Page 50: The author's email id: akshukla2006@gmail.com; shukla@sscu.iisc.ernet.in