Correlations in Dice Throwing Experiment

We consider three waiting type experiments involving throws of an unbiased die and study the correlation coefficients among the involved random variables.

Introduction

Probability theory is an interesting subject because it discusses many experiments which lead to two or more apparently correct solutions and all look convincing and hence these are paradoxes. Many such paradoxes can be found in books on probability theory (Feller [1], Mood and Graybill [2]). In what follows, we discuss a classroom encounter. A question was put to a class of students regarding three different dice throwing experiments. The discussion that followed was quite interesting. We first discuss the three experiments. We denote by \(X_{1i}\) and \(X_{2i}\) (\(i = 1, 2, 3\)), the number of 'ones' and 'twos' obtained in the \(i\)th experiment.

Experiment 1

An unbiased die is thrown till we get a SIX.

Experiment 2

An unbiased die is thrown till we get a FIVE or SIX.

Experiment 3

An unbiased die is thrown till we get a FIVE and SIX at least once.

In the above experiments \((X_{11}, X_{21})\), \((X_{12}, X_{22})\) and \((X_{13}, X_{23})\) are pairs of non-negative integer random variables. Let \(r_1, r_2\) and \(r_3\) be the correlation coefficients such that \(r_i = \text{corr. coeff. } (X_{1i}, X_{2i}), i = 1, 2, 3\).

The students were asked to comment on \(r_1, r_2\) and \(r_3\).

Keywords
Correlation coefficients, throw of a die.
They quickly responded that all the three correlation coefficients will be equal or nearly equal with the reasoning that \( P(\text{one}) = P(\text{two}) = 1/6 \) every time. However when the three \( r \)'s were evaluated, they were surprised to find that the three values of \( r \) obtained were not even nearly equal!

We give below the joint distributions of \((X_1, X_2)\) in the above mentioned experiments.

Experiment 1:
\[
p(x_1, x_2) = 1/3[(x_1 + x_2)!/x_1!x_2!](1/3)^{x_1+x_2}
\]

Experiment 2:
\[
p(x_1, x_2) = 1/2[(x_1 + x_2)!/x_1!x_2!](1/4)^{x_1+x_2}
\]

Experiment 3:
\[
p(x_1, x_2) = 2[(x_1 + x_2)!/x_1!x_2!][(1/3)^{x_1+x_2} - (1/4)^{x_1+x_2}]
\]

In each case \(x_1\) and \(x_2\) are non-negative.

We derive the joint distribution of \((X_1, X_2)\) for the first experiment. A similar reasoning can be used to obtain the joint distributions of the variables involved in the other two experiments.

**Joint Distribution of \((X_1, X_2)\) for Experiment 1**

Let \(X_1\) denote the number of 'ones', \(X_2\) denote the number of 'twos' and \(X_3\) denote the number of 3,4 and 5 obtained till the termination of the experiment (We note that the experiment stops as soon as we get a SIX).

Using multinomial distribution, we can obtain the joint distribution of \(X_1, X_2, X_3\) as follows:

\[
p(x_1, x_2, x_3)
= [(x_1 + x_2 + x_3)!/x_1!x_2!x_3!](1/6)^{x_1}(1/6)^{x_2}(3/6)^{x_3}(1/6),
\]
\[
x_1, x_2, x_3 \geq 0
\]
The marginal distribution of \((X_1, X_2)\) is given by, using binomial theorem

\[
p(x_1, x_2) = \sum_{x_3} p(x_1, x_2, x_3) \\
= \frac{[(x_1 + x_2)!/x_1!x_2!](1/6)^{x_1+x_2+1}}{\sum_{x_3} [(x_1 + x_2 + x_3)!/(x_1 + x_2)!x_3!] (3/6)^{x_3}} \\
= \frac{[(x_1 + x_2)!/x_1!x_2!](1/6)^{x_1+x_2+1}}{1 + \left( \frac{1}{3} \right) + \left( \frac{3}{6} \right)^2 + \cdots}
\]

Thus for experiment 1, we have,

\[
p(x_1, x_2) = 1/3[(x_1 + x_2)!/x_1!x_2!](3/6)^{x_1+x_2}.
\]

We omit the other algebraic details and give important results in the following table.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E(X_1) = E(X_2))</td>
<td>1</td>
<td>1/2</td>
<td>3/2</td>
</tr>
<tr>
<td>(V(X_1) = V(X_2))</td>
<td>2</td>
<td>3/4</td>
<td>11/4</td>
</tr>
<tr>
<td>(E(X_1X_2))</td>
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<td>1/2</td>
<td>7/2</td>
</tr>
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<td>(r)</td>
<td>1/2</td>
<td>1/3</td>
<td>5/11</td>
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Acknowledgement

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Suggested Reading
