Correlations in Dice Throwing Experiment

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We consider three waiting type experiments involving throws of an unbiased die and study the correlation coefficients among the involved random variables.

Introduction

Probability theory is an interesting subject because it discusses many experiments which lead to two or more apparently correct solutions and all look convincing and hence these are paradoxes. Many such paradoxes can be found in books on probability theory (Feller [1], Mood and Graybill [2]). In what follows, we discuss a classroom encounter. A question was put to a class of students regarding three different dice throwing experiments. The discussion that followed was quite interesting. We first discuss the three experiments. We denote by X_{1i} and X_{2i} ($i = 1, 2, 3,$), the number of 'ones' and 'twos' obtained in the ith experiment.

Experiment 1

An unbiased die is thrown till we get a SIX.

Experiment 2

An unbiased die is thrown till we get a FIVE or SIX.

Experiment 3

An unbiased die is thrown till we get a FIVE and SIX at least once.

In the above experiments (X_{11}, X_{21}), (X_{12}, X_{22}) and (X_{13}, X_{23}) are pairs of non-negative integer random variables. Let r_1, r_2 and r_3 be the correlation coefficients such that $r_i =$ corr. coeff. $(X_{1i}, X_{2i}), i = 1, 2, 3.$

The students were asked to comment on r_1, r_2 and r_3.

Keywords
Correlation coefficients, throw of a die.

They quickly responded that all the three correlation coefficients will be equal or nearly equal with the reasoning that $P(one) = P(two) = 1/6$ every time. However when the three r's were evaluated, they were surprised to find that the three values of r obtained were not even nearly equal!

We give below the joint distributions of (X_1, X_2) in the above mentioned experiments.

Experiment 1:

$$p(x_1, x_2) = 1/3[(x_1 + x_2)!/x_1!x_2!](1/3)^{x_1+x_2}$$

Experiment 2:

$$p(x_1, x_2) = 1/2[(x_1 + x_2)!/x_1!x_2!](1/4)^{x_1+x_2}$$

Experiment 3:

$$p(x_1, x_2) = 2[(x_1 + x_2)!/x_1!x_2!][(1/3)^{x_1+x_2} - (1/4)^{x_1+x_2}]$$

In each case x_1 and x_2 are non-negative.

We derive the joint distribution of (X_1, X_2) for the first experiment. A similar reasoning can be used to obtain the joint distributions of the variables involved in the other two experiments.

Joint Distribution of (X_1, X_2) for Experiment 1

Let X_1 denote the number of 'ones', X_2 denote the number of 'twos' and X_3 denote the number of 3,4 and 5 obtained till the termination of the experiment (We note that the experiment stops as soon as we get a SIX).

Using multinomial distribution, we can obtain the joint distribution of X_1, X_2, X_3 as follows:

$$p(x_1, x_2, x_3)$$

$$= [(x_1 + x_2 + x_3)!/x_1!x_2!x_3!](1/6)^{x_1}(1/6)^{x_2}(3/6)^{x_3}(1/6),$$

$$x_1, x_2, x_3 \geq 0$$

The marginal distribution of (X_1, X_2) is given by, using binomial theorem

$$\begin{aligned} p(x_1,x_2) &= \sum_{x_3} p(x_1,x_2,x_3) \\ &= [(x_1+x_2)!/x_1!x_2!](1/6)^{x_1+x_2+1} \\ &\quad \sum_{x_3} [(x_1+x_2+x_3)!/(x_1+x_2)!x_3!]\,(3/6)^{x_3} \\ &= [(x_1+x_2)!/x_1!x_2!](1/6)^{x_1+x_2+1} \end{aligned}$$

$$\left[1+\binom{x_1+x_2+1}{1}\left(\frac{3}{6}\right)+\binom{x_1+x_2+2}{2}\left(\frac{3}{6}\right)^2+\quad\right]$$

$$\begin{aligned} &= [(x_1+x_2)!/x_1!x_2!] \\ &\quad (1/6)^{x_1+x_2+1}[1/(1-(3/6))^{x_1+x_2+1}] \\ &= [(x_1+x_2)!/x_1!x_2!](1/3)^{x_1+x_2+1} \end{aligned}$$

Thus for experiment 1, we have,

$$p(x_1,x_2) = 1/3[(x_1+x_2)!/x_1!x_2!](1/3)^{x_1+x_2}.$$

We omit the other algebraic details and give important results in the following table.

	Experiment		
	1	2	3
$E(X_1) = E(X_2)$	1	1/2	3/2
$V(X_1) = V(X_2)$	2	3/4	11/4
$E(X_1X_2)$	2	1/2	7/2
r	1/2	1/3	5/11

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Suggested Reading

[1] W Feller, *An Introduction to Probability Theory and its Applications*, Vol. I, III edition, John Wiley & Sons, 2000.

[2] A M Mood, F A Graybill and D C Boes, *Introduction to the Theory of Statistics*, III edition, Tata McGraw Hill, 2001.