Foundations of Basic Geometry

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Part 1. Foundation of Basic Arithmetic, *Resonance*, Vol.11, No.1, January 2006.
Part 2. Foundation of Basic Arithmetic, *Resonance*, Vol.11, No.2, February 2006.

The basic notions of length, area and volume were not alien to the prehistoric civilizations. The pyramids, palaces and great baths built more than 4000 years ago provide ample evidence. We begin our investigation of geometry with a discussion of areas of simple geometric objects.

1. Areas of Rectangles and Triangles

A *rectangle* is a plane figure formed by four straight lines meeting at right angles as shown in *Figure* 1.

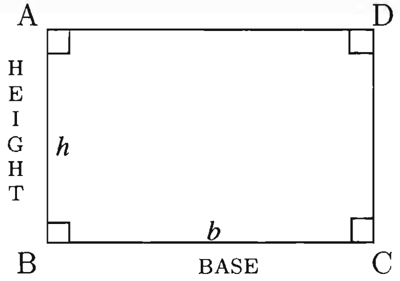


Figure 1.

We define its area as:

$$\text{area of rect ABCD} = \text{base} \times \text{height} = bh.$$

This is the most obvious way to define area for if we double the base, the area should double, or if we triple the height the area should also triple. In modern language, we say that the area is linear in each variable b and h. Up to a constant $c > 0$, which depends on the units of measurements, this is the only definition of the area if we want it to have these obvious properties.

Keywords
Area, proportionality, Pythagorean triplet.

From this definition we can show that the area of a triangle is given by the formula

$$\text{area of triangle} = \frac{1}{2}(\text{base} \times \text{height}).$$

We consider three possible cases.

1) The easiest case is that of a right triangle $\triangle$ ABC (B = 90^o). Its area is actually half the area of the rectangle ABCD (*Figure* 2).

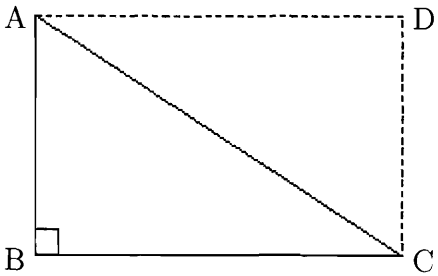


Figure 2.

Thus

$$\begin{aligned} \text{area of } \triangle ABC &= \frac{1}{2}(\text{area of the rectangle ABCD}) \\ &= \frac{1}{2}(\text{base} \times \text{height}). \end{aligned}$$

2) To compute the area of an acute triangle $\triangle$ABC, we begin by dividing the triangle into two right triangles as shown in (*Figure* 3).

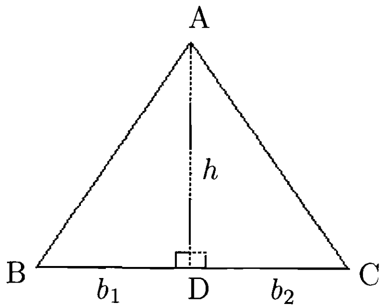


Figure 3.

The base of $\triangle$ABC $= b_1 + b_2$. Hence

$$\begin{aligned} \text{area of } \triangle\text{ABC} &= \text{area of } \triangle ABD + \text{area of } \triangle\text{ADC} \\ &= \frac{1}{2}\, b_1 h + \frac{1}{2}\, b_2 h = \frac{1}{2}\, h(b_1 + b_2) \\ &= \frac{1}{2}\, \text{height} \times (\text{base of triangle ABC}). \end{aligned}$$

So we have the same formula for the area of a triangle.

3) Finally the case of an obtuse triangle $\triangle$ ABC (*Figure* 4).

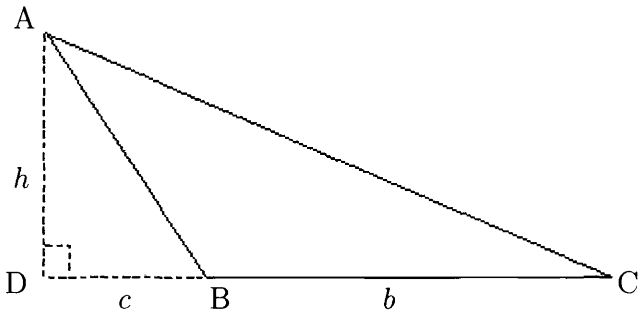


Figure 4.

Now $\triangle$ABC = $\triangle$ADC $-$ $\triangle$ADB. From this and the fact that $\triangle$ ADC and $\triangle$ ADB are right triangles,

$$\begin{aligned} \text{area of } \triangle\text{ABC} &= \text{area of } \triangle\text{ADC} - \text{area of } \triangle\text{ADB} \\ &= \frac{1}{2}\, h(b + c) - \frac{1}{2}\, hc = \frac{1}{2}\, bh. \end{aligned}$$

2. Area of a Circle

The formula for the area of a circle was known to the Chinese and the Babylonians as half of the radius times the circumference.

Next we compute the area of a circle. The formula for the area of a circle was known to the Chinese and the Babylonians as half of the radius times the circumference. For this we cut a circle into a large (even) number of equal slices and then place them as shown in *Figure* 5.

If n is very large, then this figure is almost a rectangle with its base equal to half of the circumference and its

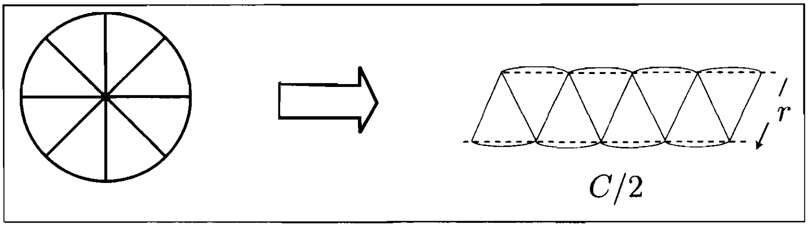


Figure 5. Changing a circle into a rectangle.

height equal to the radius. So, using the formula for the area of a rectangle, we get the formula

$$\text{area of circle } = \frac{1}{2} \text{ circumference} \times \text{radius.} \qquad (1)$$

It should be noted that one cannot claim the two sides of (1) to be exactly equal without having some idea of 'limit' Thus these ancient civilizations were not unaware of the concept of limit.

To investigate this formula further we must discuss the proportionality of the sides of similar triangles (triangles of the same shape). If we have two similar triangles with sides (x, y, z) and (X, Y, Z), as illustrated in *Figure* 6, then

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \text{ constant} = c, \text{ say}$$

or

$$\begin{aligned} X &= cx \\ Y &= cy \\ Z &= cz, \end{aligned}$$

i.e., the sides of a triangle are proportional to the sides of any triangle similar to it.

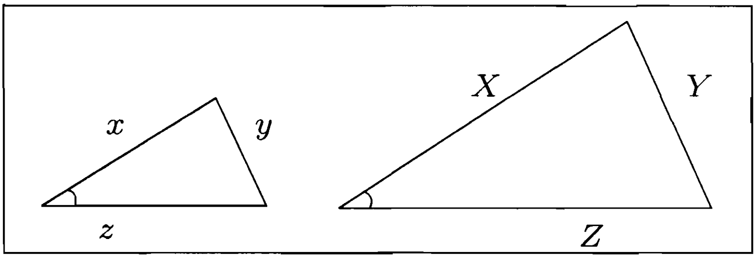


Figure 6. Similar triangles.

The ratio circumference/ diameter is the same for all circles.

Even though the proportionality of sides of similar triangles must have been obvious to the Egyptians from building the pyramids, the proof appeared much later in Euclid's *Elements*, Book 6, Proposition 4.

It is not hard to deduce the well-known fact that all circles bear the same ratio of the circumference to the diameter, i.e.,

Theorem 1. *The ratio circumference/diameter is the same for all circles.*

Proof. Let us consider any two (concentric) circles of different sizes. If C, R and c, r denote the circumference and the radius of the bigger and smaller circle respectively, then it is enough to show that

$$\frac{C}{R} = \frac{c}{r}.$$

To approximate the circumferences, we inscribe regular n-gons inside the circles as shown in (*Figure* 7) $n = 8$.

We denote by C_n and c_n the perimeters of the polygons so formed, each of n equal sides. As illustrated in *Figure* 7, the perimeter approximates the circumference of each circle. But $c_n = n \cdot b_n$ and $C_n = n \cdot B_n$, where b_n and B_n are the side lengths of the smaller and larger polygons respectively.

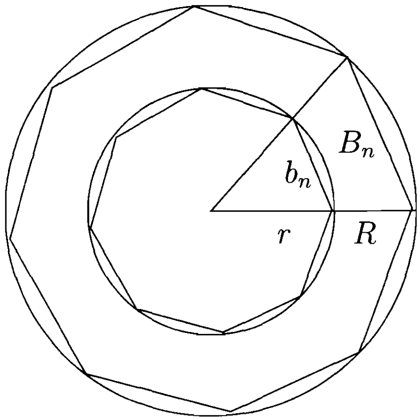


Figure 7.

By the proportionality of sides of similar triangles, it follows that

$$\frac{c_n}{C_n} = \frac{nb_n}{nB_n} = \frac{b_n}{B_n} = \frac{r}{R}.$$

Thus

$$\frac{c_n}{C_n} = \frac{r}{R}.$$

Since $\frac{c_n}{C_n}$ is independent of n (it is always equal to the constant $\frac{r}{R}$) and since c_n approaches c and C_n approaches C as n increases, it follows that

$$\frac{c}{C} = \frac{r}{R}, \quad \text{i.e.,} \quad \frac{c}{r} = \frac{C}{R}.$$

Q.E.D.

Although the first occurrence of the symbol π for this ratio appeared in 1706 in a book by William Jones, like many other symbols, use of π was popularized by the Swiss mathematician Leonhard Euler (1707-1783), probably to initialize the word *perimeter.*

Today we use the Greek letter π (pi) to denote this universal constant: any circle's circumference divided by its diameter. But the Greeks did not use this symbol. Although the first occurrence of the symbol π for this ratio appeared in 1706 in a book by William Jones, like many other symbols, use of π was popularized by the Swiss mathematician Leonhard Euler (1707–1783), probably to initialize the word *perimeter.*

From this fact and the Chinese–Babylonian formula (3) for the area of a circle, we can easily deduce the well-known formula (famously proved by Archimedes) for the area of a circle with radius r, i.e.,

$$\text{area of a circle of radius } r = \pi r^2$$

Proof. Since π is by definition the ratio circumference/diameter, we see that for a circle of radius r, its

$$\begin{aligned} \text{circumference} &= \frac{\text{circumference}}{\text{diameter}} \text{ diameter} \\ &= \pi(2r) \\ &= 2\pi r. \end{aligned}$$

So by (1),

$$\begin{aligned} \text{area} &= \frac{1}{2} \text{ circumference} \times \text{radius} \\ &= \frac{1}{2} \;\; 2\pi r \;\; r = \pi \;\; r^2 \end{aligned}$$

Q.E.D.

3. Pythagorean Theorem

This is a theorem central to all of geometry. There is the Fundamental Theorem of Algebra, the Fundamental Theorem of Arithmetic, and the Fundamental Theorem of Calculus. If there is a Fundamental Theorem of Geometry, it is the Pythagorean Theorem. Artifacts reveal that this theorem was known to the Babylonians and the Egyptians sometime around 2000 BC, and to the Chinese and Indians around 1500 BC. In fact, the oldest known proofs (or rather hints for proofs) of the Pythagorean Theorem go back to the Chinese and the Indians. However, the theorem is often referred to as the Pythagorean Theorem after the Greek mathematician Pythagoras (around 6th century BC) who was the first to prove it rigorously, but his proof was more likely using similar triangles. The proof by Euclid in Book 1 of his *Elements* is a masterpiece of mathematics writing.

Among many known proofs, the one given by the Indian mathematician Bhaskara (1114–1185 AD) looks very easy, but assumes that the sum of three angles of a triangle is 180°

Theorem. (Pythagoras) *Given a right triangle with sides a, b and hypotenuse c, we always have $a^2+b^2=c^2$.*

Proof. Bhaskara's proof was geometric, but it can be explained as follows. Cut a square of each side $a+b$ into four equal triangles and a square as shown in *Figure* 8.

By using the formulas for the area of a rectangle and triangle, we see that $(a+b)^2 = 4(\frac{1}{2}ab) + c^2$. And by

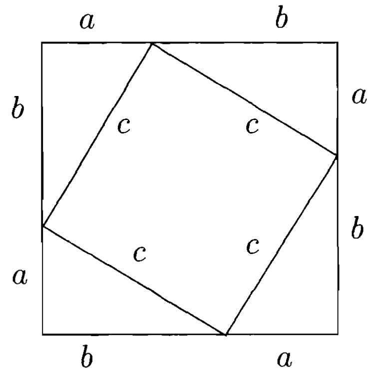


Figure 8.

simple algebra this reduces to

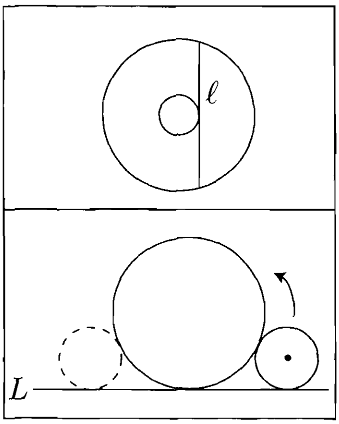
$$a^2 + b^2 = c^2$$

Q.E.D.

Exercises

1. If A, B, C with $0 < A \leq B < C$ are the areas of equilateral triangles drawn on the sides of a right triangle, show that $A + B = C$.
2. Suppose a chord of length ℓ is tangent to the inner of the two concentric circles. Compute the area of the region bounded by the chord and minor arc of the larger circle cut off by the chord (*Figure* 9).
3. As shown in *Figure* 10, circles of radii 6 and 2 are both tangent to line L and tangent externally to each other. Suppose the smaller circle rolls along the circumference of the larger one until it is again tangent to L, as shown. How far has the center of the smaller circle travelled?

Figure 9 (top).
Figure 10(bottom).



4. Pythagorean Triplets

A triplet (x, y, z) with x, y, and z all positive integers satisfying the equation $x^2 + y^2 = z^2$ is called a *Pythagorean triplet*. Note that $x^2 + y^2 = z^2$ if and only if $(nx)^2 + (ny)^2 = (nz)^2$ for any whole number n. We call the Pythagorean triplet representing the smallest right triangle of a given shape a *primitive triplet.* One such triplet, (3, 4, 5) has been known since antiquity. Other commonly recognized triplets are (5, 12, 13) and (8, 15, 17). The order of x, y is immaterial, but we shall write the odd one first. (Both cannot be even.) The following theorem describes them completely.

Theorem. *All the primitive Pythagorean triplets* (x, y, z), *i.e., integers* x, y, z *representing the sides of a right triangle (among similar ones of smallest size) are of the form*

$$\begin{aligned} x &= a^2 - b^2 \\ y &= 2ab \\ z &= a^2 + b^2 \end{aligned} \tag{2}$$

where a, b *are integers such that*

(i) $a > b > 0$;

(ii) a, b *are of opposite parity, i.e., one is odd, and the other even;*

(iii) a, b *have no common factor, other than 1.*

Before we proceed with a proof, let us find some Pythagorean triplets (x, y, z) by using this theorem.

Example. Let $a = 2$ and $b = 1$. Using equations (2) we get

$$\begin{aligned} x &= 2^2 - 1^2 = 3 \\ y &= 2\ \ 2\ \ 1 = 4 \\ z &= 2^2 + 1^2 = 5, \end{aligned}$$

Next we take $a = 3$ and $b = 2$. Our triplet (x, y, z) is now $(5, 12, 13)$ for

$$\begin{aligned} x &= 3^2 - 2^2 = 5 \\ y &= 2\ \ 3\ \ 2 = 12 \\ z &= 3^2 + 2^2 = 13, \end{aligned}$$

By choosing two numbers a and b as above, we can generate up to proportionality all right triangles with sides whose lengths are whole numbers. Thus there are infinitely many primitive Pythagorean triplets. Now that we have seen how useful the theorem is, we sketch a proof.

Proof of the Theorem. To begin we write the equation $x^2 + y^2 = z^2$ in the form

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1, \qquad (3)$$

so that we can see that finding the Pythagorean triplet is equivalent to finding points (X, Y) on the circle

$$X^2 + Y^2 = 1 \qquad (4)$$

with coordinates $X = \frac{x}{z}$ and $Y = \frac{y}{z}$ as rational numbers in the lowest form. (A *rational number* is a ratio of two whole numbers such as $\frac{1}{2}$, $\frac{3}{5}$, $\frac{22}{7}$).

From analytic geometry, we know that the circle given by (4) can be parameterized as (see Exercise 1)

$$\begin{aligned} X &= \frac{t^2 - 1}{t^2 + 1} \\ Y &= \frac{2t}{t^2 + 1}. \end{aligned} \qquad (5)$$

To get points with rational coordinates, we plug $t = \frac{a}{b}$ where a, b are integers in the lowest form into (5) and solve for X and Y:

$$X = \frac{(\frac{a}{b})^2 - 1}{(\frac{a}{b})^2 + 1} = \frac{a^2 - b^2}{a^2 + b^2}$$

and

$$Y = \frac{2\,(\frac{a}{b})}{(\frac{a}{b})^2 + 1} = \frac{2ab}{a^2 + b^2}.$$

Recall that $X = \frac{x}{z}$ and $Y = \frac{y}{z}$. Since $t = \frac{a}{b}$ is in the lowest form, so are X and Y above. Hence

$$\begin{aligned} x &= a^2 - b^2 \\ y &= 2ab, \\ z &= a^2 + b^2 \end{aligned} \tag{6}$$

It is easy to see that a and b must be as stated. Q.E.D.

Recall that all Pythagorean triplets are obtained from the primitive ones (x, y, z) as (nx, ny, nz) for n in $\mathbb{N}$.

Exercises

4. Derive the parameterization (5) for the circle (4). *Hint:* Intersect (4) with the family of lines $X = tY - 1$ through the fixed point $(-1, 0)$ on (4). In other words substitute $X = tY - 1$ in (4) and solve for Y and X in terms of t, the slope of the line $X = tY - 1$ of this family as shown in *Figure* 11.

5. Find a, b which on substitution in (2) gives $x = 12,709$, $y = 13,500$ and $z = 18,541$.

Remarks

1. The sophistication of the Babylonian mathematics is remarkable. During the period 1900-1600 BC, the Babylonians compiled tables of Pythagorean triplets (see [1], pp.8-10]). It is hard to believe that they stumbled upon

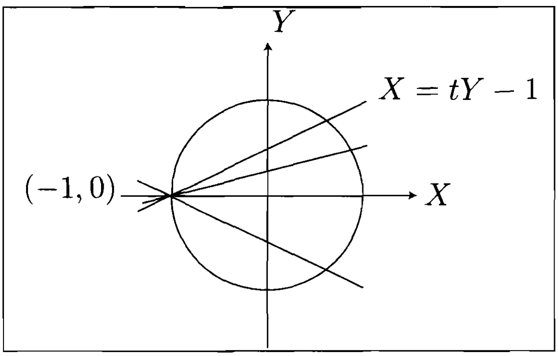


Figure 11.

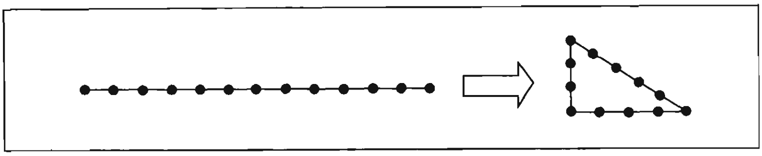


Figure 12. The Egyptians created right triangles from ropes divided into 12 equal segments..

the primitive Pythagorean triplets like $x = 12,709$, $y = 13,500$, $z = 18,541$. Plausibly, they were aware of the algorithm provided by the theorem for generating the primitive Pythagorean triplets.

2. It is believed (but there is no historical evidence) that as far back as 5000 years ago the Egyptian rope-stretchers used the Pythagorean triplet $(3,4,5)$ to construct right angles such as for pyramid building. In order to construct a triplet $(3,4,5)$, a rope was divided by eleven knots into twelve equal segments. (Since the size of the rope is arbitrary, the Egyptians must have known the proportionality of similar triangles.) The rope was laid and stretched on the ground to make a right triangle with sides 3, 4, and 5 as shown in *Figure* 12.

Suggested Reading

[1] André Weil, *Number Theory – An Approach through History*, Birkhäuser, 1983.

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The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn.

Sir Isaac Newton
(1643 to 1727)