Truchet Tilings and their Generalisations

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Tilings and patterns dominate our visual and material world. Cyril Stanley Smith was acutely sensitive to this in every aspect of his work. In a diversion he rediscovered the Truchet tilings of 1704 and added to their richness and variety.

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1. Introduction

The range and depth of Cyril Stanley Smith's erudition was astonishing. Apart from being one of the most creative metallurgists of the twentieth century, he was also an eminent historian of science and technology. The aesthetic aspects of science and the relationship between art and science interested him immensely [1]. It was this latter interest that led him to the rediscovery of a remarkable work of 1704 on tiling patterns by Sebastian Truchet and to see some of the deeper implications of Truchet's idea.

2. Sebastian Truchet and his Tilings

Sebastian Truchet was a Carmelite priest who, incidentally, was also the inventor of the 'point system' for indicating the sizes of type faces (fonts) – now familiar to all PC users!

His method of generating an infinite variety of tiling designs by the combinatorial manipulation of four-letter codes is a surprisingly early beginning for the idea of encoding visual patterns (see [2]).

Truchet's idea is simple, but its suggestive power and hidden possibilities for generalisation are immense. The Truchet tile is a square tile with a simple diagonal two-colour decoration (*Figure* 1). In a tiling of the plane by Truchet tiles, each tile can be placed in four possible orientations, which may be called A, B,

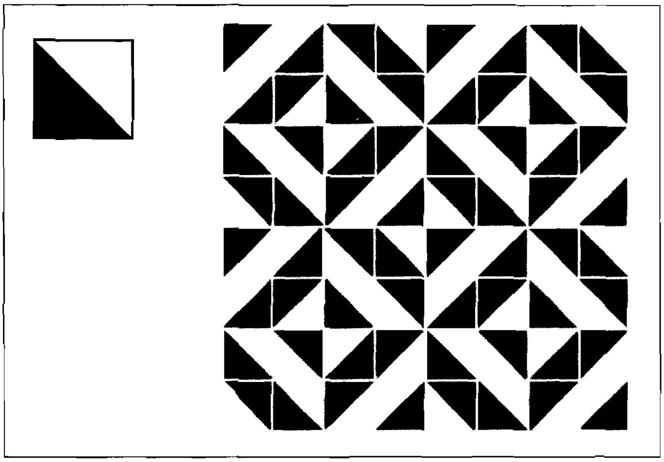

Figure 1 (left). The basic Truchet tile.
Figure 2 (right). Four unit cells of a simple Truchet tiling pattern.

C and D. *Figure* 2, for example, shows four unit cells of the pattern encoded as

BDCA...
DBAC...
ACDB...
CABD...

Truchet's publication contains a wide variety of examples. This one representative example will suffice here – many more can be seen by executing a Google search for "Truchet tiling".

Symmetry: Apart from translations, a periodic pattern may have 2-fold, 3-fold, 4-fold or 6-fold symmetries, meaning that the pattern is unchanged when it is rotated about a point through 180, 120, 90 or 60 degrees, respectively.

It may also have reflection symmetries or glide symmetries. A glide is a reflection in a line followed by a translation in the direction of the line.

The possible symmetry of a periodic Truchet tiling pattern can be any one of twelve of the seventeen 'wallpaper groups' (i.e. those that have no threefold or sixfold symmetry). An interesting exercise in combinatorics would be the enumeration and classification of Truchet patterns with a given symmetry and given number of tiles per unit cell. Clearly, twofold and fourfold symmetries, reflection and glide symmetries of a pattern correspond to combinatorial properties of its generating code. A Truchet pattern may also exhibit 'two-colour' symmetries, when the black and the white regions of the pattern are congruent, as in *Figure* 2.

"There are few places where the approaches of the artist and the scientist intersect more intimately than in the production and analysis of tiling patterns."
– *C S Smith*

Truchet's work emerged at a time when eminent mathematicians such as Fermat, Leibnitz and Pascal were developing the theory of probability, and the associated mathematics of permuta-

Zeitgeist: Indication of the spirit and mood of a particular period in history.

tions and combinations, and can be regarded as a manifestation of this *Zeitgeist*. In his article on Truchet tilings in *Leonardo*, Cyril Stanley Smith had this to say [3]: "Truchet's treatise is of considerable importance for it is in essence a graphical treatment of combinatorics, a subject that, under the influence of Pascal, Fermat and Leibniz, was at the forefront of mathematics at the time. Truchet says that he got the idea when he saw a supply of tiles for paving apartments in a château near Orléans."

The combinatorial aspects of Truchet's idea were further elaborated by Douat, whose book, published in 1722, presumably introduced the idea to a wider audience. The full title of Douat's book was *Methode pour faire une infinité de desseins différents avec des carreax mi-partis de deux couleurs par une Ligne diagonale, ou observations du P. Dominique Douat Religieux Carme de la Province de Toulouse sur un mémoire inseré dans l'histoire de l'Academie Royale des Sciences de Paris l'Année 1704, Presente par R. P. Sebastien Truchet Religieux du même ordre, Acadamicien honoraire*. They went in for lengthy book titles in those days! The art historian E H Gombrich rediscovered this obscure publication and reproduced some of its pages in *The Sense of Order* [4].

3. Variations on a Theme

In his article in *Leonardo* Cyril Stanley Smith introduced two alternatives to the basic Truchet tile. By omitting the black and white colouring and retaining only the diagonal line gives a tile with only two possible orientations, instead of four; the resulting patterns can be encoded in a *binary* notation. The resulting

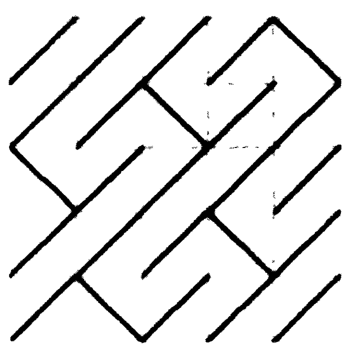

Figure 3. One of C S Smith's variants of the Truchet tiling patterns. Patterns of this kind can be specified by a binary encoding.

patterns have an intriguing 'maze-like' appearance (*Figure* 3). Whereas Truchet and Douat considered only periodic patterns, Smith's interest lay in extending the method to illustrate principles of *hierarchical* structure, which can be generated by iterative rules applied to the binary encoding. Another variant of the Truchet tile introduced by Smith is decorated by two circular arcs. This tile also has just two possible orientations and produces patterns with a curious sinuous structure (*Figure* 4).

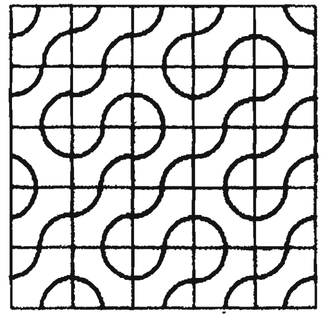

Figure 4. Another of C S Smith's variants.

Random Truchet tilings have been considered by Pickover [5]. A portion of such a random pattern built from Smith's variant of the Truchet tile, is shown in *Figure* 5. A characteristic of patterns built from this tile is the way the sinuous line partitions the plane into two regions; this has been emphasised by the grey shading of the figure. In this way, binary sequences are translated into visual patterns which can reveal interesting aspects of chaotic behaviour. Observe the occurrence of small circles in the pattern. Pickover deduced that, if the tile orientations are truly random, the number of circles divided by the number of tiles should be approximately 0.054. Similarly, the fraction of 'dumbbells' (two coalesced circles) is 0.0125. More complex statistical

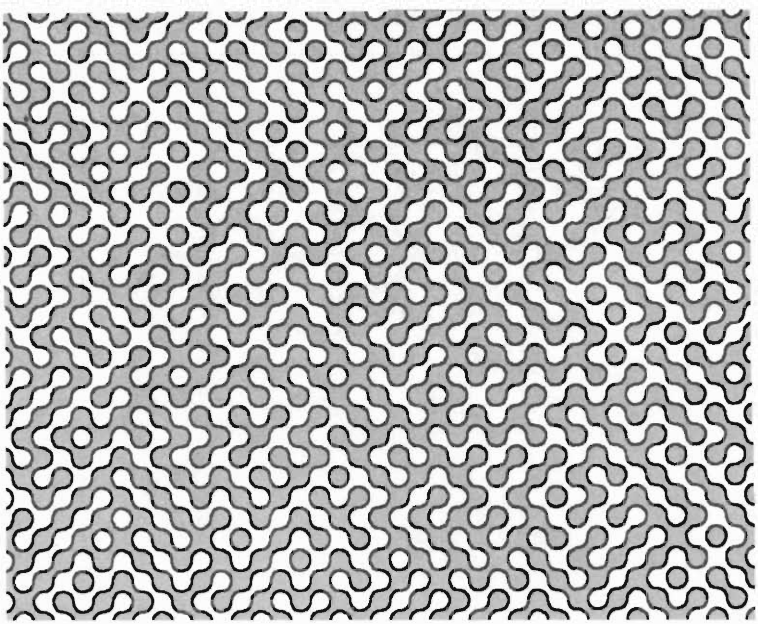

Figure 5. The appearance of a random tiling by the tiles shown in Figure 4.

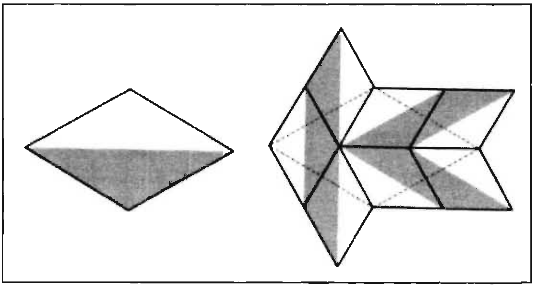

Figure 6. A single decorated rhombic tile and an 'inflation rule' for generating aperiodic patterns.

aspects of the random patterns are also calculable. Pickover proposed the use of Truchet tilings to suggest tests for true random behaviour, and for identification of subtle deviations from randomness in chaotic systems.

An interesting modification of the Truchet tile is the 60° rhombus on the left in *Figure* 6. This permits an infinite variety of tiling patterns with threefold and sixfold symmetries. In a tiling of the plane with these rhombic tiles, the individual tiles can occur in six different orientations. On the right in *Figure* 6 is a group of six of the tiles, illustrating an *inflation rule* that will generate *quasiperiodic* patterns (the inflation concept in the context of Penrose tilings is described in detail by Grünbaum and Shephard [6]). *Figure* 7 shows the patch of the tiling obtained after four iterations.

Periodic Patterns: Patterns that cover a plane may be periodic or aperiodic. A doubly periodic pattern has a unit cell (i.e. a tile) that repeats by translation, in two directions, without change of orientation, to produce the whole pattern. A periodic pattern may have other symmetries, apart from translations. Doubly periodic patterns in a plane can be classified by their symmetries. There are just seventeen types. The 17 groups of symmetries are called 'Wallpaper groups'. The Penrose tilings are aperiodic.

4. Three-Dimensional Generalisations

A question that naturally comes to mind is: are there three-dimensional analogues of the Truchet tile. We give two examples of '3D Truchet patterns'. Observe how the shape in the box in *Figure* 8 – shown in its *four* possible orientations –

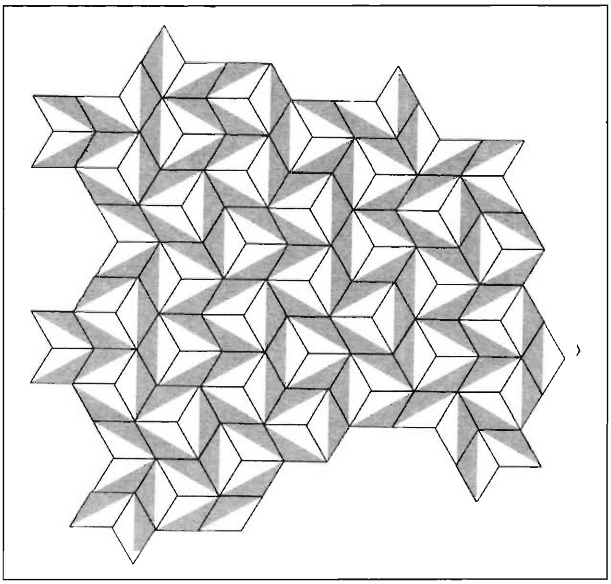

Figure 7. The appearance of the tiling produced by four iterations of the inflation rule.

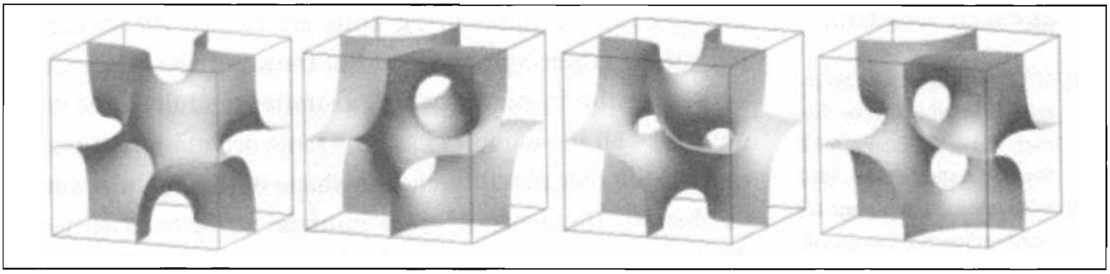

Figure 8 The four possible orientations of a three-dimensional generalisation of a Truchet tile.

meets each face of the box in a pair of arcs *like the decoration on Smith's variant of the Truchet tile*. We get an infinite variety of surfaces by tiling three-dimensional space with these 'tiles.' However, natural restrictions arise that have no analogue in the two-dimensional case, where tiles could be oriented completely arbitrarily. Because the surface patches in two contiguous boxes must match on the interface, there are in fact *only two* possible orientation relationships between a pair of contiguous tiles: they are related either by a translation or by a reflection in the common face. Moreover, if two contiguous tiles are related by reflection in their common plane, then the plane must be a reflection plane for all the pairs of tiles having their common face in that plane. An example of a portion of a 3D Truchet tiling of this kind is shown in *Figure* 9.

The curved surface patch in the basic tile has the shape of the modular unit forming one eight of a unit cell of the minimal surface known as 'Schoen's batwing' (see [7] and Ken Brakke's

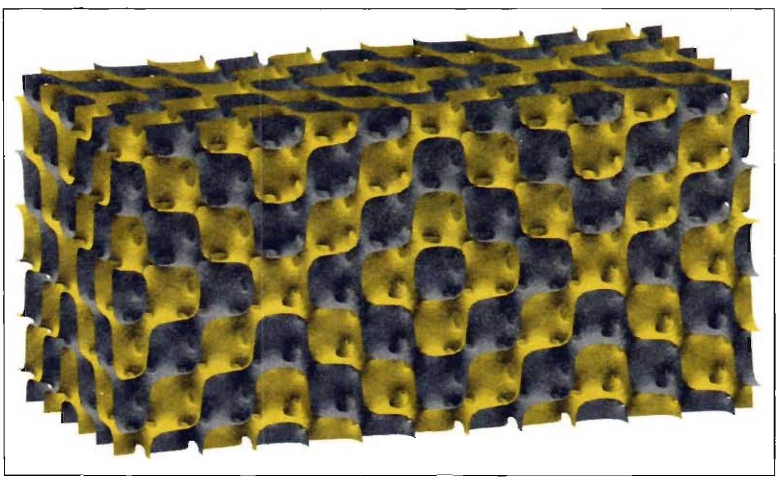

Figure 9. A surface produced by tiling three-dimensional space with the tile shown in Figure 8.

wesbsite [8]). Neighbouring units are related everywhere by reflection in their common face. For the minimal surface named by Brakke the 'pseudobatwing', a similar module is the whole unit cell – all the modules have the same orientation. It is quite remarkable that the difference in shape of the modular surface patches for these two distinct minimal surfaces, with totally different space group symmetries, is imperceptible. *Figure* 9 was produced by Alan Mackay, who first realised that the 'batwing' module can be treated as a 3D Truchet tile, thus indicating the possibility of an infinite number of possible minimal surfaces based on the 3D Truchet tiling principle.

Another possible generalisation of Truchet's idea to the description of 3D structures is its application to the description and classification of 3D weaving patterns. 3D weaving has recently become of increasing importance in the manufacture of composite materials, in which fibres embedded in a matrix constitute a reinforcement of the material. A 2D weaving pattern has two orthogonal thread directions (warp and weft). Three dimensional analogues can be constructed from a modular unit containing portions of *three* orthogonal threads (*Figure* 10). Derck van Schuylenburch has investigated the possibilities for 3D weaving patterns, with three orthogonal thread directions, subject to the condition that every two-dimensional layer is a simple 2D under-over-under-over-weave (a 'mat'). He holds patent rights to these 3D weaving methods. Contiguous units like *Figure* 10 can be related by rotation about an axis in a cube face, by screw transformations, by glides, reflections, etc. However, the requirement that the resulting pattern shall consist of stacks of simple 2D mats, in all three directions, is severely restrictive. Indeed, neighbouring cubes can then be related only by inversion in their common face or by 2-fold screw transformations (respectively O and I in van Schuylenburch's notation). In terms of a three-letter symbol listing the kind of transformation to be applied along the three orthogonal directions, we get four weaving patterns OOO, OOI, OII and III (III existing in a right-handed form IIIR or a left-handed form IIIL, according to

Suggested Reading

[1] C S Smith, *Search for Structure: Selected Essays in Science, Art and History*, MIT Press, Cambridge MA, 1981

[2] S Truchet, Memoir sur les combinaisons, *Memoires de l'Academie Royale des Sciences*, pp.363-72, 1704.

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[7] E A Lord and A L Mackay, Periodic minimal surfaces of cubic symmetry, *Curr. Sci.*, Vol.85, pp.346-62, 2003.

[8] Ken Brakke, http://www.susqu.edu/brakke/

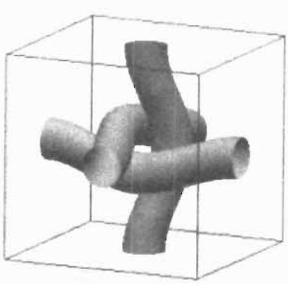


Figure 10. The basic three-dimensional tile for generating 3D weaving patterns.

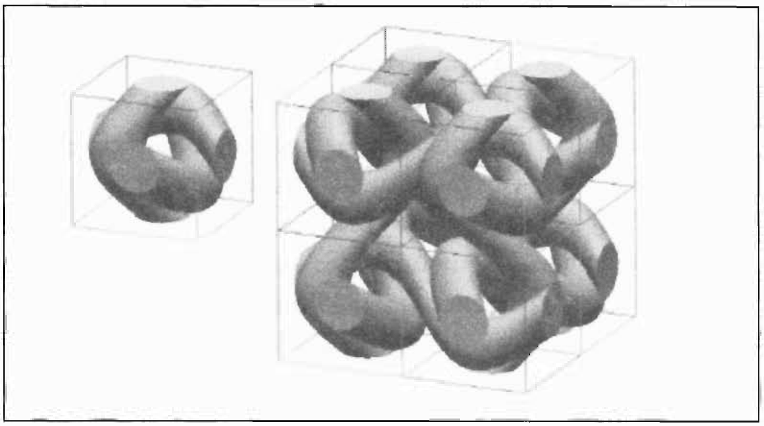


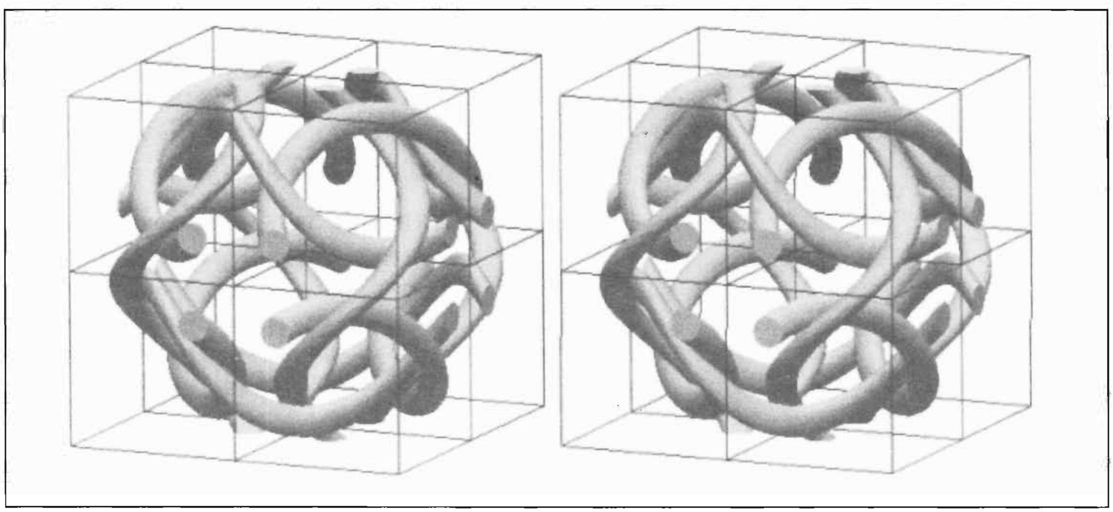
Figure 11. A 3D weave obtained by stacking the cubic tiles of Figure 10 so that contiguous tiles are related by inversions.

whether the basic module is the one shown in *Figure* 10 or its mirror image.) *Figure* 11 illustrates a unit cell of OOO (in which the threads have sinusoidal form, and *Figure* 12 shows a stereo image of IIIR, in which all the threads are right handed helices.

5. Conclusions

The advent of high speed computers has changed the way scientists think about the structure of materials. Not only has the era of computers opened up the possibility of solving problems that had previously been intractble – new, hitherto unthought of problem areas have been opened up. The idea of encoding a pattern or structure in two or three dimensions as a

Figure 12. A stereo pair of images of a 3D weave in which the basic tiles are related by a screw transformation.



string of alphanumeric symbols, and the reconstitution of a simulation of the pattern or structure from its abstract code - an 'inorganic gene' is an important instance of this change of viewpoint, likely to become increasingly important, particularly in the materials sciences. Truchet's idea, though simple, was remarkably prophetic; it is to the credit of Cyril Stanley Smith that he recognised its potential. It is our hope that the possibilities for further generalisations of Truchet's idea, that we have presented here, may act as a stimulus to further explorations.

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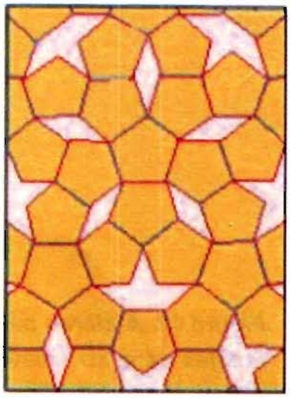

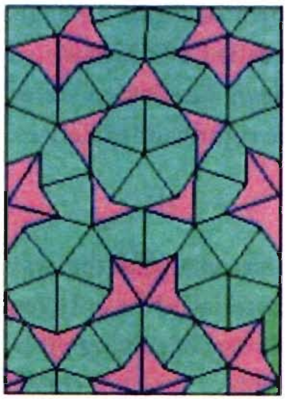

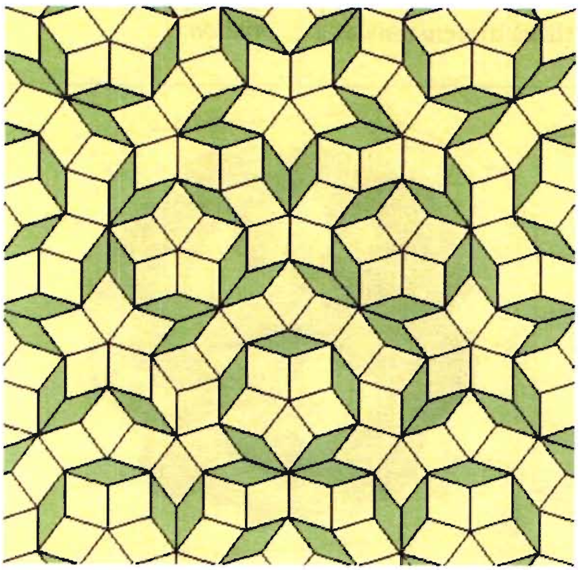

Penrose's Aperiodic Tilings

Roger Penrose's first aperiodic set of tiles – a pentagon, a star, a hexagon and a 'boat' together with a set of 'matching rules', can tile the whole plane, but a *periodic* pattern is not possible. The question of whether such an 'aperiodic set' is possible has a long history. Later, Penrose introduced two alternative aperiodic patterns, each with only *two* tiles the 'kite and dart patterns' and the 'rhomb' patterns.

All these tilings have a long-range orientational order with fivefold symmetry but no translational symmetry, and have played an important role in the developement of our understanding of quasicrystals.

Suggested Reading

[1] R Penrose, The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.*, Vol.10 pp. 266-71, 1974.

[2] E A Lord, S Ranganathan and U D Kulkarni, Tilings, coverings, clusters & quasicrystals, *Curr. Sci.*, Vol.78, pp. 64-72, 2000.