Soap Bubbles and Crystals

Jean E Taylor

Jean Taylor is Professor Emerita at Rutgers University and currently a Visitor at the Courant Institute, NYU. She works on problems related to soap bubble froths, crystals, and how they evolve under various physical laws. Much of her recent research has been interdisciplinary, joint work with materials scientists, particularly a group at NIST (National Institute for Standards and Technology).

The Plateau problem is a famous problem in the shape of bubbles. The author's solution to this problem by the introduction of geometric measure theory is described. The context of Cyril Stanley Smith's work on the shape of grains and crystals is introduced.

I first met Cyril Smith in January, 1973 during the year I was an Instructor at MIT with a brand-new mathematics PhD. I had recently proved Plateau's rules for bubbles, a century-old conjecture, so I announced a series of lectures on soap bubbles. Cyril Smith and John Cahn noted and discussed the announcement, John came to my lectures, John introduced me to Cyril, and the entire later direction of my career was formed. Cyril and John were part of a group that called themselves the Philomorphs, lovers of shapes, and I have regarded myself as a fellow philomorph ever since.

This article is based on a public lecture given on October 19, 2005, at the Indian Academy of Sciences. A similar lecture, given at the Royal Canadian Institute in November 2005, was video taped and the video and accompanying PowerPoint presentation can be viewed at http://www.psych.utoronto.ca/~rci/0005.html.

In September 1970, when I was about to enter graduate school at Princeton University (after earning master's degrees in physical chemistry and in mathematics), I heard a beautiful lecture by Princeton professor Fred Almgren on the subject of Geometric Measure Theory. In talking to him after the lecture, I learned that: (1) Suppose you put three half circles on the unit sphere, with endpoints meeting at the north and south poles, at equal angles to each other. It could be proved that the surface consist-

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ing of 3 half-disks meeting along the line inside the sphere from the north to the south pole is the least area surface with those thee half circles as boundary. This is indeed the shape of the soap film which forms on a wire frame of three such half-circles. (2) If you wiggled even just one of the half circles, as smoothly as you liked, and as little as you liked, then there was no proof that the triple junction line would even stay of finite length.

"Surface tension phenomena have attracted the interest of many mathematicians and physicists. According to Maxwell, Leonardo da Vinci is credited with the discovery of capillary action ... Isacc Newton was the first to produce the idea of molecular forces affecting the cohesion of liquids, and Laplace, Gauss, and Poisson all worked on the theory of how the surface energies arose...The culmination of the thermodynamic approach was the work of J W Gibbs."

Jean E Taylor
The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces, *Annals of Mathematics*, Vol.103, pp.489-539, 1976.

I was amazed that such a problem could still be open. The reason for the difficulty lay in what was meant by a "surface" and by "the boundary of a surface." In calculus, a surface is defined as the graph of a function, or more generally as the image of a piece of a plane under a function taking that piece into 3-space (also called a mapping). The boundary of the surface is the image of the boundary of the piece. So right away one needs a formulation in which the triple junction does not count as part of the boundary. That isn't too difficult; remembering that the normal direction of a surface determines the orientation of its boundary (right hand rule), you simply work modulo 3. Thus if the three half circles are oriented from the north to the south pole, the three half disks have as extra boundary 3 times the oriented line from the south to the north pole. But 3=0 modulo 3, and therefore the triple line does not count as being part of the boundary.

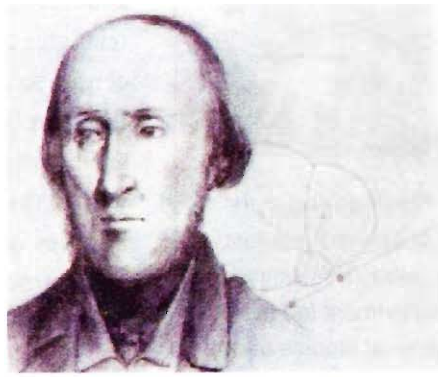
But other problems also arise. A typical way to try to obtain a minimizing surface is to take the limit of a sequence of mappings which have areas converging toward the infimum of all mappings with the same boundary. But without some controls on mappings, terrible things can happen. For example, with the boundary simply being a circle, a disk with lots of very thin long fingers sticking up from it can have an area only a little more than the area of the flat disk, and a minimizing sequence could have as limit the disk with one dimensional hairs extending to, say, every point in 3-space with rational coordinates. On the other hand, if you require the surfaces in the minimizing sequence to be smooth enough to avoid that problem, then with a general boundary curve you might find yourself with too many conditions to be able to guarantee that there is a limit. This problem is difficult enough that one of the first Fields medals was given to Jesse Douglas in the 1930's for his proof of existence of a smooth minimizer for a mapping of a disk with any given smooth boundary. And in 1970 the problem had not been solved for triple junctions.

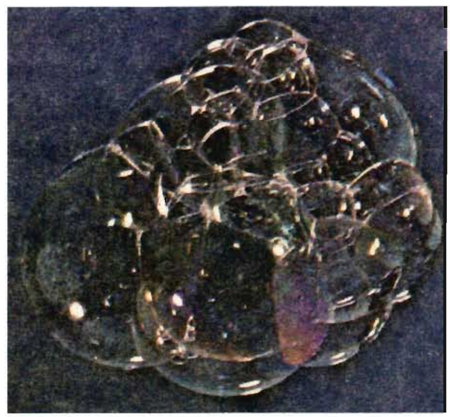
In mathematics, when something does not exist, often you invent a new class of things in which it does exist. For example, there is no rational number (fraction p/q with p and q whole numbers, $q \neq 0$) which is the square root of 2. The solution is to create a new type of number, which by definition has the desired property of its square being 2. Even better, fill in all the "holes" in the number line, to get all real numbers. As another example, there is no real number whose square is -1. The solution is to add another type of number, called i, and define it to have the property that $i*i = -1$. Even better, go on to define complex numbers. Similarly, there was no proof of existence, nor any structure theorems, for modeling soap films and soap bubble clusters. The solution was define a new subject, 'geometric measure theory', in which existence is easy. Then once you have such objects, you prove what you can about regularity, singularity structure, etc. In geometric measure theory, you define a surface by where its area is. Then the limit of those hairy disks is just a flat disk; you do not "see" all the limit 0-radius tubes because they do not have any area. This is the heart and soul of geometric measure theory – geometric, because it deals with shapes and surfaces, measure because you define the surface by measuring the area of where it is.

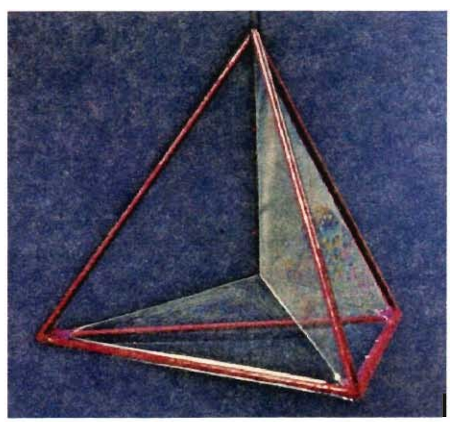
When I arrived at Princeton, it was known that in the class of surfaces called flat chains modulo 3 (which are oriented surfaces, defined in terms of where their area is, with boundary counted modulo 3), area-minimizers with a given boundary exist and consist of smooth pieces of surface except for a singular set. There was nothing known about the singular set except that its area is zero. What I proved in my PhD thesis was that the singular set had to consist of smooth curves along which smooth

Joseph Plateau, Jean Taylor and Soap Bubbles

Frederick J Almgren and Jean Taylor have written in an evocative fashion about soap bubbles. "Soap bubbles and soap films evoke a special fascination. Their shifting iridescence, their response to a puff of air, their fragility – all contribute to their charm. More captivating still is their perfection of geometry." While the subject has attracted attention for centuries, the first scientist to have devoted attention to this subject was the Belgian physicist Joseph Plateau (*Figure* 1). In 1873 he published a two-volume work summarizing his decades of research into soap bubbles and related phenomena due to surface tension. He gave the rules governing the geometry of bubbles, *without any proof.* It is a remarkable achievement as these experiments were performed when Plateau was blind. In a cluster of soap bubbles (*Figure* 2) the films meet in one of two ways: either three surfaces meet at 120-degree angles along a curve; or six surfaces meet at a vertex, forming angles of about 109 degrees (*Figure* 3).







In a leap of imagination, Cyril Stanley Smith drew in 1948 an analogy between soap films and metal grains and arrived at important conclusions concerning the shape of metal grains and their growth. Until Jean Taylor came along in the mid-1970's, Plateau's patterns were just a set of empirical rules. As a follow-up to her doctoral thesis, she was able to prove that "Plateau's rulės were a necessary consequence of the energy-minimizing principle – no other yet unobserved configurations were possible – thus settling a question that had been open for more than a hundred years."

Smith gazing at the soap films in a test tube. This simple experiment led him to seminal studies on the shape of metallic grains.

surfaces meet three at a time at equal angles. In particular, in that original problem, the triple junction did indeed have to be of finite length. The same modulo–3 trick cannot be used for general soap films. In particular, there is no way to orient the six triangles that run from the edges of a regular tetrahedron into its center so that each triple junction has all three edges oriented the same way. Rather, another (much more complicated) class of surfaces from geometric measure theory has to be invoked. And within this class, I managed to prove that Plateau's rules for soap bubbles apply: smooth pieces of surface meet three at a time along smooth triple junction curves or six at a time at isolated points where four such triple junction curves come together (as the smooth image of that collection of six triangles from the edges to the center of the regular tetrahedron).

What does this have to do with Cyril Smith and crystals? Metals are made up of many small crystals, called grains, within which the atoms are regularly ordered. The disorder of one grain abutting against another gives rise to an extra energy, the surface free energy. Each pair of abutting grains has its own surface free energy function, which varies with the normal direction of the interface as well as with the misorientation of the crystals. If the dependence on normal direction is negligible, then the surface free energy becomes proportional to the surface area. Although there is little reason to believe the interfaces actually minimize free energy, they may come close enough that it is somewhat reasonable to model them as soap bubble clusters. Cyril Smith was able to photograph the grain boundaries in an aluminum-tin alloy and demonstrate the similarity.

Throughout my career, I wrote a number of papers investigating energy-minimizing surfaces where the energy depends so strongly on normal direction that the equilibrium shapes are completely facetted. Then I investigated crystal growth problems, in both 2 and 3 dimensions. My current research, with John Cahn, concerns the fact that during grain growth, crystals may rotate as well as grow or shrink. I wish Cyril were here to be able to appreciate it.

Suggested Reading

[1] **Frederick J Almgren Jr and Jean E Taylor, The geometry of soap films and soap bubbles, *Scientific American*, Vol. 235, p.82, 1976.**

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