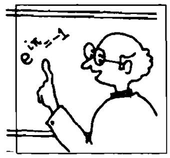

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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A Divine Surprise: The Golden Mean and the Round-off Error

This article deals with an example of the systematic round-off error that can be encountered in numerical computations. The example is based on the recursion relation used for calculating higher powers of the golden mean. In the process, the link between the golden mean and the Fibonacci sequence also becomes apparent.

A friend of mine was teaching computational methods to the final year undergraduates last year and conveniently decided to be ingenuous (and perhaps lazy?) about setting up the final examination. All his departmental colleagues were asked to contribute one or two small problems encountered routinely in the course of scientific computation. And I reckoned it would not be unfair to throw something at the current generation of students which our teachers saw fit to hit us with!

Keywords
Golden mean, machine accuracy, Fibonacci sequence.

This is a problem dealing with the accuracy of numerical computations. We need to remember that computers do not perform mathematical operations to infinite accuracy. In fact, a computer is capable of storing a floating-point number only to a fixed number of decimal places. For every type of computer, there is a characteristic number known as the *machine accuracy* (denoted by ε_m). This is defined as the smallest number which when added to unity produces a floating point number different from unity (see *Box* 1]. Therefore, we can think of ε_m as the fractional error in any arithmetic operation, commonly known as the *round-off error*.

If the round-off error is random then the maximum total error in performing n arithmetic operations would be

Box 1. Machine Accuracy

The value of ε_m depends on how many bytes the computer hardware uses to store floating-point numbers. For Pentium-IV models, the value of the smallest double precision (64-bit) floating point number is $\varepsilon_m = 2.22\times10^{-16}$. The corresponding ε_m for single precision (32-bit) floating-point number is 1.19×10^{-7}. It is easy to write a small program, as given below, to find this number on a given machine.

```
c
c       Program   : Machine Accuracy
c       Language  : Fortran77
c       Precision : Double (64-bit)
c
        program epsilon_m
        implicit none
        integer n,npts
        parameter(npts=100)
        real*8 s,t
c
        s = 1.d0
        do n = 1,npts
           s = 5.d-01*s
           t = s + 1.d0
           if (t.le.1.d0) exit
        end do
        s = 2.d0*s
        print*,n,s
c
        stop
        end
```

$\sim \sqrt{n}\varepsilon_m$. However if there is a systematic accumulation then the error can grow to very large values. A beautiful example of this is encountered if the n-th power of the *golden mean* defined by

$$\phi_c = \frac{\sqrt{5}-1}{2} \simeq 0.61803398, \qquad (1)$$

is computed using the recursion relation,

$$\phi_c^{n+1} = \phi_c^{n-1} - \phi_c^n. \qquad (2)$$

So, the students were asked to show that the recursion relation is unstable on a computer. On a typical 32-bit machine the relative error using the recursion relation becomes larger than 1 around $n \sim 16$. If the computation is done on double precision then n is ~ 38.

A golden rectangle having sides in the ratio ϕ is defined such that partitioning the original rectangle into the largest square and a new rectangle would result in the new rectangle having sides again in the ratio ϕ.

I considered my job done after dutifully handing this problem over to my friend. Alas! Soon it was grading time and the students had not only shown the obvious but had also came up with an explanation for this behaviour linking the recursion relation to the Fibonacci Sequence. This indeed was worth checking. Internet, the greatest company on a lonely Saturday evening, instantly threw up a plethora of information relating the golden mean and the Fibonacci sequence (see *Box* 2).

Golden Mean

The golden mean, also known as the *divine proportion*, *golden ratio* or *golden section*, is defined as,

$$\phi = \frac{1}{2}(1+\sqrt{5}). \qquad (3)$$

Notice that this ϕ is different from ϕ_c defined earlier. In fact, $\phi_c = \phi - 1$ and is sometimes known as the *conjugate golden mean*. Presently, we shall see how ϕ and ϕ_c are intimately related, particularly via the above-mentioned recursion relation (and some).

Box 2. Fibonacci Sequence

Leonardo Fibonacci discovered a simple numerical series while studying the population growth of rabbits and published his results in Liber Abaci in the year 1202. Starting with 0 and 1, each new number in this series is simply the sum of the two preceding it. The first few terms of the sequence are given by,

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \tag{2a}$$

Surprisingly, the ratio of each successive pair of numbers in the series asymptotically approaches ϕ, the golden mean. Perhaps it is not so surprising then that the analytical formula for obtaining the n-th term in the Fibonacci series can be written in terms of ϕ and ϕ_c as following,

$$f(n) = \{\phi^n - (-\phi_c)^n\}/\sqrt{5}. \tag{2b}$$

Many of the simple geometric forms, like pentagram, decagon, dodecagon, have ϕ as one of the inherent ratios. For example, the ratio of the circum-radius to the length of the side of a decagon, given by $\frac{1}{2}\csc(\pi/10)$ equals ϕ. A *golden rectangle* having sides in the ratio ϕ is defined such that partitioning the original rectangle into the largest square and a new rectangle would result in the new rectangle having sides again in the ratio ϕ. Successive points dividing a golden rectangle into squares lie on a logarithmic spiral (see *Figure* 1). The legs of a *golden triangle* (an isosceles triangle with a vertex angle of 36°) are in a golden ratio to its base and, in fact, this was the method used by Pythagoras to construct ϕ.

It is not very clear exactly when ϕ became known to mankind. Perhaps it has been discovered again and again by different civilisations at different points of time.

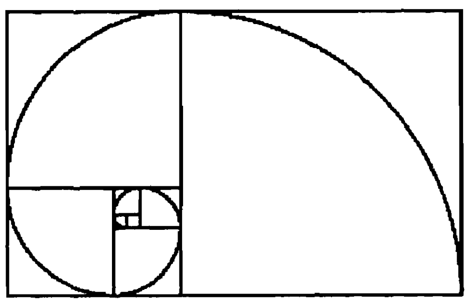

Figure 1. A 'golden rectangle' showing the progression of the 'logarithmic spiral. The logarithmic spiral is given by the polar equation, $r = ae^{b\theta}$, where a, b are constants.

Box 3. Golden Masterpieces

Apparently the most aesthetically pleasing ratio, $1 : \phi$, have fascinated mankind from time immemorial. Da Vinci (circa 1500 AD) called it the *sectio aurea* (Latin for 'golden section') and showed how various parts of the human anatomy are related to each other through this ratio. One of the most famous artworks is Da Vinci's *The Last Supper* in which he has used ϕ to define all the fundamental proportions of the picture. The ubiquitous presence of ϕ has been found even in the works of the great musicians like Mozart and Beethoven. Beethoven's Fifth (also known as the 'Emperor') concerto seems to have been sectioned precisely at ϕ points. The modern generation has not been immune either. Sergei Eisenstein divided his classic silent film *The Battleship Potemkin* using golden section points to start important scenes in the film, measuring these by lengths on the celluloid.

Certainly, the ancient Egyptians and Greeks knew about ϕ having used this in some of their greatest architectural marvels (Great Pyramid, Parthenon). In fact, ϕ is named after the Greek sculptor Phidias (circa 5th century BC) who appears to have made extensive use of ϕ in his work. Euclid referred to dividing a line at the 0.6180399.. point, as dividing a line in the 'extreme and mean ratio' in his *Elements*. Many of the renaissance artists and even musicians used ϕ to create some of their immortal masterpieces (see *Box* 3).

Staircase Approximations

The recursion relation, talked about earlier, reduces to

$$x^2 = 1 - x \tag{4}$$

in the limit of $n = 1$. It is easy to see that the roots of this quadratic equation are nothing but ϕ_c and $-\phi$. We can now play some more games with these two numbers. Let us look at all possible quadratic equations with roots having magnitudes equal to ϕ and ϕ_c. There are four equations in all, with the following pair of roots: $\{\phi, -\phi_c\}$, $\{-\phi, \phi_c\}$, $\{-\phi, -\phi_c\}$ and $\{\phi, \phi_c\}$. The equations corresponding to these are,

Many of the renaissance artists and even musicians used ϕ to create some of their immortal masterpieces.

$$(x - \phi_c)(x + \phi) = 0 \Rightarrow x^2 + x - 1 = 0 \Rightarrow x = -1 + \frac{1}{x}, \tag{5}$$

$$(x+\phi_c)(x-\phi) = 0 \Rightarrow x^2-x-1 = 0 \Rightarrow x = 1+\frac{1}{x}, \quad (6)$$

$$(x-\phi_c)(x-\phi) = 0 \Rightarrow x^2-\sqrt{5}x+1 = 0 \Rightarrow x = \sqrt{5}-\frac{1}{x}, \quad (7)$$

$$(x+\phi_c)(x+\phi) = 0 \Rightarrow x^2+\sqrt{5}x+1 = 0 \Rightarrow x = -\sqrt{5}-\frac{1}{x}. \quad (8)$$

Notice the last relation corresponding to each equation. The interesting aspect of these is the way x can be expressed in terms of $1/x$. For example, if we take the second equation and keep substituting x by $(1 + 1/x)$ the following sequence emerges,

$$x = 1+\frac{1}{x} = 1+\frac{1}{1+\frac{1}{x}} = 1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$$
$$= 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}} = 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}}}\cdots \quad (9)$$

In other words, we have generated a staircase (or *continued fraction* if you wish to be formal) approximation to x. Obviously, after a few terms the initial value of x one started with in the right hand side becomes irrelevant and the final value always approaches ϕ asymptotically. The most amusing part of this game is the fact that the staircase approximation for x from all the four equations converge to either ϕ or $-\phi$ (depending on which is the corresponding root for the relevant quadratic equation). The reason these staircases never pick up ϕ_c has to do with the fact that $\phi_c < 1$ whereas $\phi > 1$ (see *Box* 4).

Recursion Relations

And finally, to the recursion relations themselves. Again, consider the one given by,

$$x^{n+1} = x^{n-1} - x^n$$

Box 4. Staircases Favor the ϕ !

Consider the staircase relation shown in equation (9). If we simplify the staircase structure the following pattern emerges,

$$\begin{aligned} x_1 &= 1+\frac{1}{x_o}=\frac{1+x_o}{x_o} \\ x_2 &= 1+\frac{1}{1+\frac{1}{x_o}}=\frac{1+2x_o}{1+x_o} \\ x_3 &= 1+\frac{1}{1+\frac{1}{1+\frac{1}{x_o}}}=\frac{2+3x_o}{1+2x_o} \end{aligned}$$

$$x_n = \ldots = \frac{f_{n+1}+f_{n+2}x_o}{f_n+f_{n+1}x_o}, \tag{4a}$$

where x_n is the value of x in the n-th step of staircase iteration and x_o is the original value we use in the right-hand side to begin the approximation. It is evident that the staircase is nothing but another way of hiding the relation between the ϕ and Fibonacci. Let us now go around this and use equation (2b) to rewrite the above relation in terms of the ϕ s again. This gives us,

$$x_n = \frac{\{\phi^{n+1}-(-\phi_c)^{n+1}\}+\{\phi^{n+2}-(-\phi_c)^{n+2}\}x_o}{\{\phi^n-(-\phi_c)^n\}+\{\phi^{n+1}-(-\phi_c)^{n+1}\}x_o}. \tag{4b}$$

As expected, x is a mixture of the two roots : ϕ and $-\phi_c$. Since, ϕ_c is smaller than unity, ϕ_c^n decreases with increasing n and becomes negligible compared to ϕ^n and we have,

$$x \simeq_{n\to\infty} \phi, \tag{4c}$$

which has, in the limit of large n, become independent of x_o and equal to ϕ itself.

Very neat. But if we look carefully, we'd find Mr. Fibonacci lurking underneath. Let's just look at the first few terms somewhat more closely then. We would have,

$$\begin{aligned} x^2 &= 1-x \\ x^3 &= x-x^2=-1+2x \\ x^4 &= x^2-x^3=2-3x \\ x^5 &= x^3-x^4=-3+5x \\ &\ldots . \end{aligned}$$

Wait a minute. The coefficients seem to be popping out of the Fibonacci sequence. Indeed they are. In fact, written formally, the recursion relation is,

$$x^n = (-)^{n-1} f_n x + (-)^n f_{n-1}$$

where f_n is the n-th term in the Fibonacci sequence. It comes as no surprise then that all the four equations above have recursion relations expressible in terms of Fibonacci numbers. It is easy to check that the recursion relations are,

$$\begin{aligned}
&(x+\phi_c)(x-\phi) = 0 \Rightarrow \quad x^2 - x - 1 = 0 \\
&\Rightarrow \quad x^n = f_n x + f_{n-1} && (10) \\
&(x-\phi_c)(x+\phi) = 0 \Rightarrow \quad x^2 + x - 1 = 0 \\
&\Rightarrow \quad x^n = (-)^{n-1} f_n x + (-)^n f_{n-1} && (11) \\
&(x-\phi_c)(x-\phi) = 0 \Rightarrow \quad x^2 - \sqrt{5}x + 1 = 0 \\
&\Rightarrow \quad x^n = \sqrt{5} f_n x - (f_{n-2} + f_n) \quad \text{for even } n && (12) \\
&\Rightarrow \quad x^n = (f_{n-1} + f_{n+1})x - \sqrt{5} f_{n-1} \quad \text{for odd } n && (13) \\
&(x+\phi_c)(x+\phi) = 0 \Rightarrow \quad x^2 + \sqrt{5}x + 1 = 0 \\
&\Rightarrow \quad x^n = -\sqrt{5} f_n x - (f_{n-2} + f_n) \quad \text{for even } n && (14) \\
&\Rightarrow \quad x^n = (f_{n-1} + f_{n+1})x + \sqrt{5} f_{n-1} \quad \text{for odd } n && (15)
\end{aligned}$$

The recursion relations are somewhat complicated for the last two equations but once again the n-th power of x is written in terms of a set of Fibonacci numbers and x itself. And herein lies the solution to the systematic accumulation of the round-off error.

Suppose the machine accuracy introduces a round-off error equal to ε_m to ϕ_c in the first step such that the numerical value is given by,

$$\phi_c^{\text{num}} = \phi_c + \varepsilon_m. \tag{16}$$

Then, as the computation progresses, the result in the n-th step would be,

$$(\phi_c^{\text{num}})^n = (-)^{n-1} f_n(\phi_c + \varepsilon_m) + (-)^n f_{n-1}, \tag{17}$$

with a total error of $f_n \varepsilon_m$. Evidently, the error in the n-th step being proportional to f_n is clearly much larger than $\sqrt{n}\varepsilon_m$. Quite naturally, the recursion relation would be completely off the mark when $\phi_c^n \sim f_n\varepsilon$. It is obvious that in single precision calculations this would happen when $n \sim 16$ and in double precision around $n = 38$.

Once again, there is an asymmetry between ϕ and ϕ_c. The error accumulated through the iterative process becomes significant only for the conjugate golden mean. Since the magnitude of ϕ_c is smaller than unity the left hand side decreases recursively (as we are raising the power at every step) whereas the error goes on increasing since it is progressing in Fibonacci series. This is different from the case of ϕ which is already larger than unity to begin with. Thankfully, n has to be really huge in order for the error to be significant in comparison to ϕ^n and we can, for all practical purposes, neglect that.

Penrose Tiles, discovered by Roger Penrose, which can be used to tile a surface in five-fold symmetry are shapes based on ϕ.

Fascinating though the history is, ϕ keeps showing up again and again as we march forward in our endeavour to understand the mysteries of the universe. *Penrose Tiles*, discovered by Roger Penrose, which can be used to tile a surface in five-fold symmetry are shapes based on ϕ. Quasi-crystals, discovered in the '80s, are materials with perfect long-range order, but with no three-dimensional translational periodicity. And the unit structure making up these quasi-crystals are some generalised Penrose tiles, again based on ϕ.

Presumably, with the progress of time ϕ would continue to excite some of our best brains and would give rise to many such wonderful discoveries. For the rest of us though, we would blindly use our credit cards without ever stopping to think why the ratio of its sides have always appeared to be just right (aesthetically speaking)!

Acknowledgments

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