Āryabhaṭa and Axial Rotation of Earth

2. Nakṣatra Dina (The Sidereal Day)

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Part 1. Aryabhata and Axial Rotation of Earth – Khagola (The Celestial Sphere), *Resonance*, Vol.11, No.3, pp.51-68, 2006.

In the first part of this series, we discussed the celestial sphere and Āryabhaṭa's principle of axial rotation; in this part we shall discuss in detail the concept of sidereal day and then mention Āryabhaṭa's computations on the duration of sidereal day.

It is unfortunate that science students in India, by and large, do not have technical awareness regarding the researches of ancient Indian scientists. Thus, although there are plenty of articles on Āryabhaṭa, their contents have remained confined to research journals and scholarly texts without percolating into the general cultural consciousness.

The original statements of Āryabhaṭa on axial rotation and sidereal day are spread over 4 verses out of his 85 verses on astronomy. It will not be possible to make a serious analysis of the entire range of Āryabhaṭa's work in a few pages. We hope that the preliminary exposure will encourage youngsters to acquire some general knowledge of astronomy and make a deeper study of Āryabhaṭa's work using existing literatures and their own independent judgements.

Rising and Setting of Stars

Recall that, due to rotation of the Earth, the so-called fixed stars appear to execute a daily revolution around the Earth. A few stars, called *circumpolar* stars, never descend below the horizon. But an observer at a place of moderate latitude (for instance, at a place in India) sees most stars rise in the east, ascend the sky in circular

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paths, set in the west, and then again appear in the east at the *same* point as before. An observer at the equator sees all (sufficiently bright) stars rise and set.

Now, if a star rises on the eastern horizon at a particular time today, it will rise again tomorrow, from the same point, but about 4 minutes earlier. It will rise about 1 hour earlier after 15 days, 2 hours earlier after 30 days, and so on. It is only after about a year (roughly 24×15 days), that the star will again rise at a time which would be close to its time of rising today. The observation also shows that during this time interval (of nearly one year), the star gains an extra orbit over the Sun, that is, the number of times the star revolves around the Earth is one more than the number of days (in the year). We shall first try to understand this phenomenon and define relevant concepts.

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Sidereal and Solar Day

A *nakṣatra dina* or a *sidereal day* is the time taken by any nakṣatra (fixed star) to perform one complete revolution around the Earth (relative to an observer on Earth). The Latin word "sidereal" means "pertaining to the (fixed) stars" The duration of a sidereal day is thus the period of one complete rotation of Earth around its axis. For non-circumpolar stars, the duration between two successive star-rises equals the length of the sidereal day.

The sidereal day differes slightly from the commonly used *civil day* comprising 24 hours. The latter is called *sāvana dina* or *solar day.* The length of a solar day is the time taken by the Sun to go around the Earth once (relative to a terrestrial observer); for instance, the duration between two successive sunrises.

The duration of a sidereal day is the period of one complete rotation of Earth around its axis. The sidereal day differes slightly from the commonly used civil day comprising 24 hours.

The slight difference between sidereal and solar day occurs because the Earth, apart from rotating around its axis, also revolves around the Sun. Let C denote the centre of the Earth and P an object fixed on the Earth's surface. Suppose that, at a certain instant, the Sun S and a fixed star X are both vertically above P. Thus, at this initial instant, C, P, S and X are collinear. After one sidereal day (when the Earth completes one full turn), X will again be vertically above P (i.e., the line CP again coincides with CX). But, by this time (of one complete rotation), the Earth has moved a considerable distance along its orbit around the Sun. The directions CS and CP now make an angle equal to the arc through which the Earth has moved along its orbit. The Earth has to rotate by this extra angle in order that CP again coincides with CS, i.e., P again sees the Sun S overhead. (See *Figure* 1.) Thus the solar day is longer than sidereal day.

The subtle distinction between the solar and sidereal day was stated by Āryabhaṭa (Kālakriyā 5):

ravibhūyogā divasāḥ, kvāvartāścāpi nākṣatrāḥ

[*ravi* : Sun; *bhū* : Earth; *yogā* : conjunction; *ku* : Earth; *āvarta* : rotation]

> *The conjunction of the Sun and the Earth forms the [civil] days; while the rotation of Earth causes the nakṣatra dina (sidereal days).*

Let us re-examine the sidereal and solar day in the celestial sphere framework (described in Part 1).

Ecliptic

The great circle on the celestial sphere formed by the intersection of the plane of Earth's orbit (around the Sun) and the celestial sphere is called the *ecliptic* (*krāntivṛtta*). The genesis of the term lies in the fact that 'eclipses' of the Sun or the Moon can occur only when the Moon passes through the Earth's orbital plane (which defines the ecliptic).

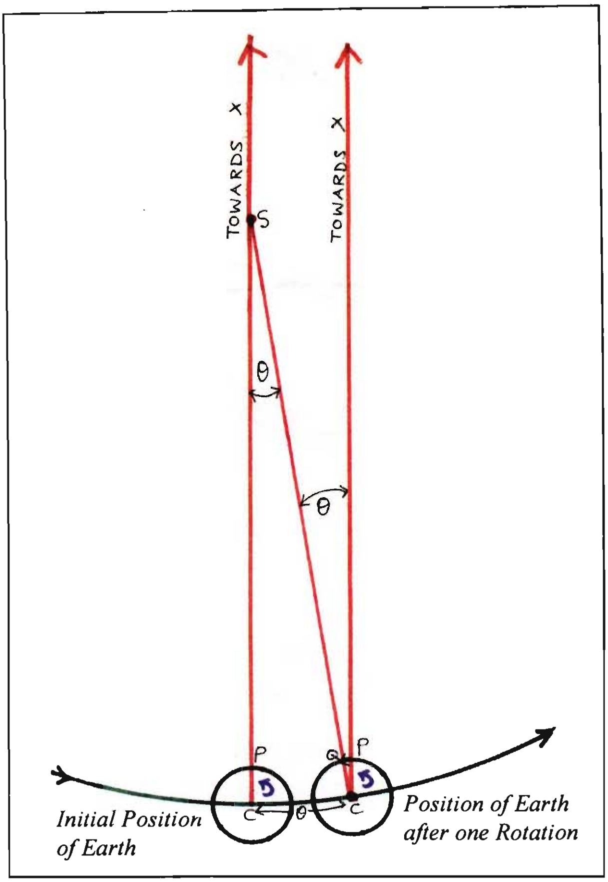

Figure 1. Sidereal and Solar day.
C: Centre of Earth
S: Sun
→ : Orbit of Earth (around the Sun)
↺ : Direction of rotation of Earth
θ : Angular distance traversed by Earth in its heliocentric orbit in one sidereal day.
Note: For convenience, the magnitude of the angle θ has been exaggerated.

The apparent motion of the Sun on the celestial sphere (as perceived by an observer on Earth) induced by the combined effect of Earth's rotation and revolution is thus composed of two movements:

(i) a daily movement from east to west parallel to the celestial equator – like that of any other star – caused by Earth's rotation from west to east.

(ii) an eastward annual movement (relative to the fixed stars) along the ecliptic caused by Earth's revolution around the Sun.

Since the Earth takes a year (roughly 365 days) to make a complete 360-degree revolution, the induced annual

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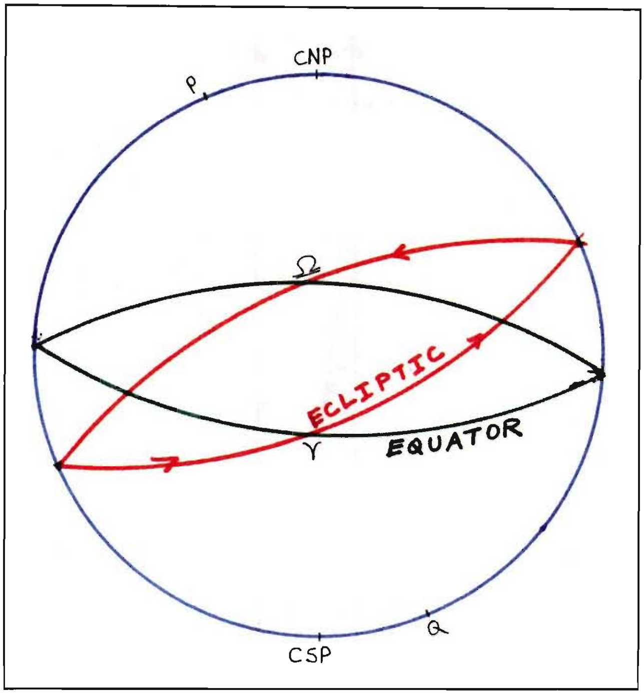

Figure 2. Ecliptic
CNP: Celestial North Pole
CSP: Celestial South Pole
P, Q: Poles of the ecliptic
γ: Vernal Equinox
Ω: Autumnal equinox
: Direction of Sun's annual path along the ecliptic.

movement of the Sun along the ecliptic takes place at the rate of about one degree per day.

As a result of factor (ii), unlike a fixed star, the Sun does not return to the same point after its daily cycle – for instance, (unlike typical star-rises) the Sun rises at different spots on successive days. We give a concrete example to illustrate the Sun's resultant path. Suppose that, at some instant, the Sun is at the point of intersection of the celestial equator with the ecliptic called the *vernal equinox* (see *Figure* 2 and *Box* 1). Its orbit for the next 24 hours will be close to the celestial equator. While factor (i) tends to keep the Sun on the celestial equator, factor (ii) tends to drag the Sun away from the equator, resulting in a continuous eastward slide. Therefore, after an orbit of 24 hours, the Sun, instead of returning to the vernal equinox, reaches a point on the ecliptic slightly to the east of the vernal equinox. The Sun then moves along a path close to a small circle parallel to the equator but sliding further away; and,

Box 1. Vernal Equinox

The vernal equinox (*mahāviṣuva*) is an important reference point in ancient as well as modern astronomy ["vernal" : "in spring"; "equinox" : "equal day and night"]. It is the point on the celestial equator which the Sun crosses during its movement along the ecliptic from south to north. (See *Figure* 2.) The other point of intersection of the ecliptic and the equator is called the *autumnal equinox*. The Sun crosses these two points on (or around) 21 March and 23 September respectively. On these dates, the day and night are of equal duration as the Sun's orbit on these dates is close to the celestial equator.

Due to precession, the vernal equinox has a slow movement (at an average annual rate of $50.27''$), relative to the fixed stars, along the ecliptic. The vernal equinox is often called the *first point of Aries* as it used to be in the Aries constellation at the time of nomenclature. (The name remains though the point is no longer in the Aries constellation.)

after 24 hours, arrives at a new point on the ecliptic further to the east. And so on.

Due to factor (ii), the Sun takes slightly more time than a fixed star to complete its daily revolution around Earth; for, while the star merely traverses parallel to the equator, the Sun has to traverse an extra distance along the ecliptic. (This is another way of understanding why a sidereal day is shorter than the solar day.)

Since the west-east deviation of the Sun (from a perfect east-west circular orbit parallel to celestial equator) takes place at the rate of about 1 degree (i.e., $\frac{1}{360}$ of a complete cycle) per day (24 hours), the difference between a sidereal and solar day is approximately ($\frac{24}{360} \times 60 =$) 4 minutes. As time is calibrated so as to make a solar day equal to 24 hours, the duration of a sidereal day becomes, roughly, 23 hours 56 minutes. (The accurate value is 23 hours 56 minutes 4.091 seconds.)

We avoided precise definitions and rigorous analyses so as to quickly convey an informal simplified picture. There are however various technical subtleties which have to be considered for obtaining accurate results. We give two examples: non-uniformity of solar day and precession.

Even Earth's rotation is not strictly uniform. There is a mild slowing of Earth's rotation due to tidal breaking. The average deceleration is roughly at the rate of 1.4 milliseconds per day per century. According to some scientists, the tsunami-causing earthquake of December 2004 has affected Earth's rotation (and shape) — it has caused the Earth to spin slightly faster which has resulted in a shortening of the day by 2.68 microseconds.

Unlike the rotation of Earth around its axis[1], the angular velocity of the Earth, in its (elliptical) orbit around the Sun, is not uniform. Consequently the Sun appears to move somewhat irregularly relative to the fixed stars. Thus the Sun traverses the ecliptic at a non-uniform rate.

Besides, the Earth's axis of rotation is inclined at an angle of about $23^0 26' 21''$ from its orbital plane. Thus the celestial equator (which determines time measurements) and the ecliptic make a considerable angle ($23^0 26' 21''$).

Due to these two factors, the duration of a solar day is not uniform. One therefore considers the *mean solar day* (*madhya sāvana dina*) which is, loosely speaking, the average duration of a solar day. (Again we avoid formal definition.) An hour is defined so that 1 mean solar day = 24 hours. (One similarly defines a sidereal hour as $\frac{1}{24}$ of a sidereal day.)

Again, because of the tilt of the Earth's axis of rotation (from the perpendicular to the plane of the Earth's orbit around the Sun), the axis performs a conical motion around the perpendicular – as in the case of a spinning top. (The analogy is only partially valid. See [2], pp.226–228.) Note that the plane of Earth's orbit determines the ecliptic and the axis of rotation corresponds to the line joining the celestial poles. Therefore, on the celestial sphere, the celestial north pole is seen to execute a circular orbit around the pole of the ecliptic. The movement is slow – it takes about 25,800 years to complete an orbit. This phenomenon of rotation of the Earth's axis, around the line joining the celestial ecliptic poles, is called *precession*[2]. The reader can now see that the sidereal day too has to be defined more carefully taking precession into account. For, precession affects the duration of the orbits of the stars especially those near the poles. Precession was called *ayana calana* in ancient Indian astronomy.

[2] Utpal Mukhopadhyay, Precession of the Equinoxes and its Importance in Calendar Making, *Resonance*, Vol.8, No.4, 2003.

In the context of the formation of solar day, we mention an interesting phenomenon. At the Earth's north pole (where the celestial equator is simply the horizon), the Sun remains above the equator (= horizon) from 21 March till 23 September, i.e., there is no sunset during the entire 6-month stretch. For the remaining 6 months, when the Sun is below the equator (= horizon), there is no sunrise. Thus, one has the peculiar phenomenon of a 6-month day and 6-month night! This fact is mentioned by Āryabhaṭa in Gola 17.

Rate of Rotation

The Earth makes a complete rotation of 360 degrees in a sidereal day. Thus, to rotate by an angle of 1 minute ($=\frac{1}{60}$ degree), the time taken by Earth is $\frac{1}{60\times 360}$ of a sidereal day – a time interval called "*prāṇa*" in Indian astronomy (see *Box* 2).

Box 2. Prāṇa as a Unit of Time

The divisions and further subdivisions of a day have been recorded in Kālakriyā 1,2. The relevant lines are:

ṣaṣṭirnāḍyo divasaḥ ṣaṣṭiśca vināḍikā nāḍī
gurvakṣarāṇi ṣaṣṭirvināḍikārkṣo ṣaḍeva vā prāṇāḥ

60 nādi = 1 day; 60 vināḍikā = 1 nādi;
60 gurvakṣara = 1 sidereal vināḍikā = 6 prāṇa.

[*ārkṣa* : sidereal] Thus 1 prāṇa = $\frac{1}{6}$ sidereal vināḍikā = $\frac{1}{6\times 60}$ sidereal nādi = $\frac{1}{6\times 60\times 60}$ sidereal day.

Thus a "prāṇa" is a precisely defined technical term for the time duration which is $\frac{1}{6\times 60\times 60}$ of a sidereal day, i.e., 1 prāṇa ($=\frac{24\times 60\times 60}{6\times 60\times 60}$) = 4 sidereal seconds.

The suggestive word "prāṇa" was chosen probably to help one form a concrete idea of the specified duration. One could mentally conceive of this unit as, roughly, the time taken (by a normal person under normal conditions) for one "respiration" (also called "prāṇa")! Similarly, the name "gurvakṣara" — a precise term for $\frac{1}{10}$ of a "prāṇa" (i.e., 0.4 sidereal second) conveys the impression that it is the time normally taken to pronounce a long syllable ("guru akṣara") in a normal condition with a moderate flow of voice.

A solar year is the period of Earth's revolution around the Sun. In fact, depending on the choice of the reference point, there are different types of solar years.

This fact was stated by Āryabhaṭa (Gītikā 6):

prāṇenaiti kalāṁ bhūḥ

[*prāṇena* : in a prāna; *eti* : rotates; *kalā* : minute of arc; *bhū* : Earth.]

In one prāṇa, the Earth rotates by an angle of one minute of arc.

But more significant is Āryabhaṭa's accurate estimate of the duration of a sidereal day. We first define the sidereal year.

Sidereal Solar Year

A solar year is the period of Earth's revolution around the Sun. For an observer on Earth, it is the time taken by the Sun to complete one orbit in its annual path on the ecliptic. The definition, so far, is ambiguous! In order to define a complete orbit, there has to be some standard of reference to be used as a starting point. In fact, depending on the choice of the reference point, there are different types of solar years.

One such reference point is given by the line joining the Sun and some specified 'fixed star'. A *sidereal solar year* or *nirayana sauravarṣa* is the time interval between two successive crossings of this line by the Earth. Equivalently, in the celestial sphere model, a sidereal solar year is the time interval between two successive passages of Sun through the *same point* on the ecliptic *relative to the fixed stars*. We clarify here that we are considering the projection along the ecliptic of the Sun's composite movement on the celestial sphere. Thus, for our present purpose, the Sun will be considered to be at a point X on the ecliptic whenever it reaches *any point* on the circle through X parallel to the celestial equator.

Note that the notion of "same point" is relative. For instance, due to precession, the 'vernal equinox', an oft-used reference point, is *not* fixed relative to the fixed stars. Consequently, if one considers the time interval between two successive passages of the Sun through the vernal equinox, one will get a a solar year different from the sidereal solar year. (It is called the *tropical year* or *sāyana sauravarṣa* [*sāyana* : with the precession].)

If one considers the time interval between two successive passages of the Sun through the vernal equinox, one will get a a solar year different from the sidereal solar year.

As the Sun completes one orbit around Earth, the fixed stars traverse a little more than one orbit in the same time duration. The difference keeps on accumulating. Since, the Sun returns to a given position relative to the fixed stars (for the first time) after a sidereal year, the cumulative difference amounts to one complete extra orbit in the year. That is, if r denotes the number of times the Sun (apparently) travels around the Earth in a sidereal year (r is not an integer), then the number of orbits of the fixed stars around the Earth, in a sidereal year, is $r + 1$. Thus, in x sidereal solar years, if the Sun makes n orbits around the Earth with reference to the fixed stars, then the stars themselves will make $n + x$ orbits around the Earth, i.e., n mean solar days $= n + x$ sidereal days.

Number of Rotations of Earth in 4320000 Years: Duration of a Sidereal Day

The Āryabhaṭīya begins with an Invocation (Gītikā 1), followed by a description of the alphabetical system of representing numbers (Gītikā 2) discussed in the appendix which follows this article. In Gītikā 3 – the very first statement on astronomy proper – Āryabhaṭa mentions:

> yugaravibhagaṇāḥ khyughṛ, śaśi cayagiyiṅuśuchṛlṛ,
> ku ṅiśibuṇlṣkhṛ prāk ...

[*ravi* : Sun; *bhagaṇa* : revolution; *khyughṛ* : 4320000; *śaśi* : Moon; *cayagiyiṅuśuchṛlṛ* : 57753336; *ku* : Earth; *ṅiśibuṇlṣkhṛ* : 1582237500 ; *prāk* : eastward]

In Aryabhata's theory, the period of rotation of Earth (equivalently, the duration of a sidereal day) = (1577917500/ 1582237500) × 24 hours; which works out to be 23 hours 56 minutes 4.1 seconds. Given that the modern value is 23 hours 56 minutes 4.091 seconds, Aryabhata's accuracy here is truly remarkable.

In a yuga, the Sun revolves 4,320,000 times, the Moon 57,753,336 times, the Earth 1,582,237,500 times eastward ...

Thus, according to Āryabhaṭa, the number of eastward rotations of the Earth (i.e., the number of sidereal days) in 4320000 sidereal solar years is 1582237500. It follows that the number of mean solar days in that period = 1582237500 – 4320000 = 1577917500. Therefore, in Āryabhaṭa's theory, the period of rotation of Earth (equivalently, the duration of a sidereal day) = $\frac{1577917500}{1582237500} \times 24$ hours; which works out to be 23 hours 56 minutes 4.1 seconds (as the reader can easily verify). Given that the modern value is 23 hours 56 minutes 4.091 seconds, Āryabhaṭa's accuracy here is truly remarkable.

Note that, by Āryabhaṭīya, a sidereal solar year consists of $\frac{1577917500}{4320000} (= 365\frac{11175}{43200})$ mean solar days which is exactly 365 days 6 hours 12 minutes 30 seconds.[3] In terms of decimal fractions, Āryabhaṭa's estimate of the duration of a sidereal solar year, upto 5 decimal places, becomes 365.25868 days. The modern estimate is 365.25636 whereas Ptolemy's value was 365.24666 ([1], p 7).

How did Āryabhaṭa arrive at the estimate of 1582237500 rotations (or sidereal days) in 4320000 years? The only clue is an obscure expression in verse Gola 48:

kṣitiraviyogād dinakṛd ... prasādhitah ...

From the conjunction of kṣiti (Earth or the horizon) and ravi (Sun) has been determined the dinakṛd (day-maker, Sun)...

The phrase "*the conjunction of the Earth and the Sun*" possibly refers to the number of sunrises in a sidereal solar year. Recall that (in *Kālakriyā* 5), it was stated

[3] To arrive at the exact figure, the reader is advised not to convert the given fraction to decimal fraction — as is done in some astronomy texts — which would introduce rounding-off errors leading to avoidable approximations. Rather convert the fraction to a mixed fraction; multiply the proper fraction part by 24; again convert the product into mixed fraction; multiply the proper fraction by 60; etc.

that due to the conjunction of the Sun and the Earth, the civil (i.e., solar) days are formed. Thus, very probably, Āryabhaṭa first made an estimate of the number of mean solar days in a sidereal solar year, arrived at an estimate equivalent to $365\frac{11175}{43200}$, whence he got the integer $(4320000 \times 365\frac{11175}{43200} + 4320000 =)$ 1582237500.

The duration of the sidereal solar year was probably determined from a regular and meticulous record of the angular distances of the Sun from some bright star and a comparative study of the data for intervals of 365 and 366 days. This distance was possibly computed by observing the time that elapsed between the risings (or settings) of a bright star and the Sun.

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As the length of a solar day is not constant, the determination of the sidereal year in terms of mean solar days would have required observations over a long period of time. Apart from his own observations, Āryabhaṭa might have had relevant data based on observations of earlier astronomers. In fact, his commentator Bhāskara I (6th century) said ([3], p.xxiv): "*The old people remember their yuga revolutions from continuity of tradition.*"

James Q Jacobs makes an interesting observation ([4]). Āryabhaṭa's estimate that 1,582,237,500 rotations of the Earth equal 57,753,336 lunar orbits gives an extremely accurate ratio $\frac{1582237500}{57753336} = 27.3964693572$ for the number of rotations of Earth per lunar orbit. It is correct up to seven decimal digits – according to modern estimates, the value in 500 CE was 27.39646514. The value for 2000 CE is 27.39646289. Curiously, the correct value around 1604 BCE was 27.39646936 which matches Āryabhaṭa's value up to ten decimal digits.

I think the history of Indian astronomy to be as a whole the most extraordinary monument of history of sciences, a very epistemology by its very self and perhaps the most enlightening knowledge of man's search for knowledge.

– *Roger Billard*

In this connection, Jacobs also points out that Āryabhaṭa's estimate for the number of days per lunar orbit is 27.321668. This matches with the correct value in his time (27.3216638) again up to seven decimal digits; and even more with the correct value in 1604 BC: 27.32166801 The value for 2000 CE is 27.32166120.

One then wonders whether Āryabhaṭa's estimate was based on an astonishingly accurate ancient Indian source dating back to 1600 BCE. Perhaps there were gifted astronomers in that remote past. (See *Box* 3.) Āryabhaṭa

Box 3. Indian Astronomers before Āryabhaṭa

As mentioned in the Chāndogya Upaniṣad (VII.1.2, 4), *nakṣatra-vidyā* (science of asterisms) was among the core disciplines of study in the Vedic era. Astronomers were called *nakṣatra-darśa* (star-observers) or *gaṇaka*. The sage Atri (who was among the originators of the oldest Vedic hymns) and his descendants were distinguised for expertise in accurate eclipse prediction and planetary astronomy. The Ṛg-Veda (V.40.5–9) describes a solar eclipse observed by Atri (dated 3928 BCE in [5], p.116; [6], pp.173–174). The Taittirīya Brāhmaṇa (III.10.9) eulogises Ahīna, Devabhāga and Śūṣa for attaining bliss due to their absorption in the science of the Sun, i.e., astronomy ([7], pp.20–21); sage Mātsya is also mentioned (I.5.2, 1) in the context of astronomy.

Vṛddha Garga is the most ancient astronomer referred to in post-Vedic treatises. The Mahābhārata (XII.59.11) refers to him as the court-astronomer of the great King Pṛthu. The epic (IX.37.14–17) mentions that a holy *tirtha* on the Sarasvatī was named after the Maharṣi as *Garga-srota* ("stream of Garga"). This was the sacred place where Vṛddha Garga performed ascetic penance for self-purification and attained mastery over astronomy. Ṛṣis of high merit and rigorous discipline used to assemble here to acquire the profound knowledge of astronomy from the venerated Ṛṣi.

Ancient Indian traditions mention a list of 18 astronomy texts called *siddhāntas* ("established theories") named after Sūrya, Pitāmaha, Vyāsa, Vaśiṣtha, Atri, Parāśara, Kāśyapa, Nārada, Garga, Marīci, Manu, Aṅgira, Lomaśa, Puliśa, Cyavana, Yavana, Bhṛgu and Śaunaka. Varāhamihira (505–587 CE), himself a prominent astronomer, also mentions Viṣṇugupta, Asita-Devala, Ṛṣiputra, Maya, Bādarāyaṇa and Nagnajit. Many of the treatises by the above astronomers got lost even by the time of Varāhamihira; none are available in their original forms.

While considerable astronomical knowledge is embedded in early Vedic literatures, the oldest available treatise devoted *exclusively* to astronomy is the *Vedāṅga Jyotiṣa* (c. 1300 BCE vide [6]) composed by sage Lagadha. The *Vedāṅga* era represents a transitional period in Indian civilisation when the Vedic culture was on the wane and there was a consequent attempt to organise and formulate the extant knowledge and systematise them into various branches called *Śāstras*.

Āryabhaṭīya (499 CE) is the earliest extant astronomy treatise after the Vedāṅga Jyotiṣa. It was composed during the "Classical Age" of post-Vedic India. Towards the beginning (Gaṇita 1) and the end (Gola 48–50) of his text, Āryabhaṭa had made a general acknowledgement of his predecessors.

himself mentions (Gola 50) that Āryabhaṭiya presents "the eternal truths of astronomy which were formerly revealed by Svāyambhu". But who was "Svāyambhu"? It is an open question.

A natural question on the verse in Gītikā 3: Why the time-scale of 4320000 years? We shall discuss it in the next part of this article. We shall also see Āryabhaṭa's principle of rotation in historical perspective.

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Suggested Reading

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[5] P C. Sengupta, *Ancient Indian Chronology*, University of Calcutta, 1947.
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Appendix

Arbyabhata's Alphabetical Coding of Large Numbers

And when the ancient Indians chose more often than not to throw whatever they wished to endure, even philosophy, science and law, into metrical form, it was not merely to aid the memory, – they were able to memorise huge prose Brahmanas quite as accurately as the Vedic hymnal or the metrical Upanishads, – but because they perceived that metrical speech has in itself not only an easier durability, but a greater natural power than unmetrical, not only an intenser value of sound, but a force to compel language and sense to heighten themselves in order to fall fitly into this stricter mould. [Sri Aurobindo in *The Future Poetry*, p 18–19.]

Texts in ancient Indian science were composed in verses and brevity was considered a desirable attribute for a scientific treatise. The richness and flexibility of the Sanskrit language were used effectively to devise terms in mathematics, astronomy and other sciences which were not only suggestive of the underlying concepts but also enabled important scientific statements to be encapsulated briefly in verse form. One serious difficulty for the authors of works on astronomy was the problem of efficient and harmonious representation of large numbers while preserving the metres of the verses. We show how Āryabhaṭa blended his linguistic and mathematical skills to find an interesting solution to this challenge.

Recall that the Sanskrit alphabet is based on the scientific phonetic principle of "one sound one symbol" and that, in a written word (as in speech), a vowel of a syllable tends to coalesce with its preceding consonant so that each syllable usually gets denoted by a single symbol. Making an ingenious use of this phonetic alphabet, Āryabhaṭa invented a novel scheme for representing large numbers with only a few letters. It was, in essence, akin to a place-value system with base hundred – the consonants of the Sanskrit alphabet denoting numerals and the vowels attached to the consonants indicating their respective place-values (in the sense of "powers of 100" and not literal positions).

Before quoting Āryabhaṭa's verse, we explain his scheme in a convenient form. We first discuss his representation of numbers up to 10^{18}. Let the vowels of the Sanskrit alphabet (in usual order) denote the powers $100^n (= 10^{2n})$ $(0 \leq n \leq 8)$ – the same value is to be attached to a short vowel and its corresponding long vowel (this step was taken probably to avoid confusion during oral transmission). Thus the assignment:

a = *ā* = 1; *i* = *ī* = 10^2; *u* = *ū* = 10^4; *ṛ* = 10^6; *ḷ* = 10^8; *e* = 10^{10}; *o* = 10^{12}; *ai* = 10^{14}; *au* = 10^{16}.

Assign to the twenty-five "*varga*" (classed) consonants from *k* to *m* the numbers (rather numerals) from 1 to 25:

k = 1; *kh* = 2; *g* = 3; *gh* = 4; *ṅ* = 5;

c = 6; *ch* = 7; *j* = 8; *jh* = 9; *ñ* = 10;

ṭ = 11; *ṭh* = 12; *ḍ* = 13; *ḍh* = 14; *ṇ* = 15;

t = 16; *th* = 17; *d* = 18; *dh* = 19; *n* = 20;

p = 21; *ph* = 22; *b* = 23; *bh* = 24; *m* = 25.

The above 25 letters are called *vargākṣara* as they are classified into 5 *vargas* (classes) – *k-varga*, *c-varga*, etc, each with 5 letters.

The syllable formed by joining a vowel with value 10^{2x} to a varga letter with value z is assigned the number $z \times 10^{2x}$. Thus, *khu* = 2×10^4; *ghṛ* = 4×10^6; *ṅi* = 5×10^2; *ṇḷ* = 15×10^8; *bu* = 23×10^4; etc. This already enables one to represent any number of the form $z \times 10^{2x}$ for $1 \leq z \leq 25$ and $0 \leq x \leq 8$ with just a single alphabetical symbol. In particular, any number of the form $z \times 10^{2x}$ $(1 \leq z \leq 9)$ or $q \times 10^{2x+1}$ $(1 \leq q \leq 2)$, where $0 \leq x \leq 8$, can be represented by a single syllable.

Now use the eight "*avarga*" (unclassed) consonants to represent a digit q, $3 \leq q \leq 10$, occurring in a number of the form $q \times 10^{2x+1}$ $(0 \leq x \leq 8)$, by assigning values:

y = 30; *r* = 40; *l* = 50; *v* = 60; *ś* = 70; *ṣ* = 80; *s* = 90; *h* = 100.

Thus *yu* = 30×10^4 $(= 3 \times 10^5)$; *śi* = 70×10^2 $(= 7 \times 10^3)$; *ṣṛ* = 80×10^6 $(= 8 \times 10^7)$.

The largest number covered so far is *hau* = 10^{18}. For numbers larger than 10^{18}, use the nine distinct vowels suitably, as above, in blocks of 18. (For instance, vowels with *anusvāra* (ṁ) could be used to denote the places between 10^{18} and 10^{36}.)

While each of the numbers from 1 to 25 and the first ten multiples of 10 can be represented by a single letter, all other numbers within 100 can be represented by two letters. Thus an n-digit number gets represented *verbally* using at most n alphabetical characters; usually much fewer symbols is needed. For instance, the two-syllabic " khyughṛ " denotes the 7-digit number $(2+30) \times 10^4 + 4 \times 10^6 = 432 \times 10^4$; while the five-syllabic " ṅiśibuṇḷṣkhṛ " denotes the 10-digit number $5 \times 10^2 + 70 \times 10^2 + 23 \times 10^4 + 15 \times 10^8 + (80+2) \times 10^6 = 1582237500$.

Since the role of "place-values" in this scheme is played by vowels and not by actual "places", a permutation of positions of the syllables does not alter the numbers (unlike the decimal place-value notation) – for instance, both *mani* and *nima* represent the same number 2025. Thus, while Āryabhaṭa's scheme extracts that aspect of the place-value notation (the idea of expressing numbers through powers of x using x numerals) which results in conciseness, it bypasses the rigidity of position inherent in a place-value scheme. Also note that numbers can have various representations – the number 30 could be denoted by the first *avarga* letter *ya* as well as by a combination of the two *varga* letters *ṅa* and *ma* [=5+25]; 43 by *rāga* [40+3] or by *naba* [20+23]. For the purpose of versification, such flexibility and scope for variations were desirable.

In Gītikā 2, the first technical verse in Āryabhaṭīya (Gītikā 1 being an Invocation), Āryabhaṭa gave a strenuously terse description of the system in the form of a rule giving a correspondence between his centesimal alphabetical notation and the decimal place value notation:

vargākṣarāṇi varge'varge'vargākṣarāṇi kāt ṅmau yaḥ

khadvinavake svarā nava varge'varge navāntyavarge vā

[*varga*: class, classed, block, square; *akṣara* : letter; *kha*: zero, void, hollow, sky; *dvi*: two, double; *nava*: nine; *svara*: vowel; *antya*: last, following; *vā*: or, and, as, like.]

The varga letters, beginning with k, [are to be used] in the varga [places], the avarga letters in the avarga [places]; [in such a way that] ṅa plus ma equals ya. The nine vowels [are to mark the eighteen] zeros formed by the nine pairs of varga and avarga [places]. A like [procedure is to be repeated using] nine [vowel-symbols] for the subsequent blocks [of eighteen varga, avarga places].

The readers would have noticed a *double entendre* in the verse. In Sanskrit, the two terms *varga* and *avarga* denote, respectively, classed and unclassed (consonants) as well as perfect squares and non-squares. Here the decimal places are also being called *varga* or *avarga* depending on whether the underlying power of ten is a perfect square 10^{2x} or is a non-square 10^{2x+1}. In each block of 18 decimal places, each of the nine distinct vowels is associated with two consecutive decimal places – one *varga* (10^{2x}) and one *avarga* (10^{2x+1}): *a* is associated with the unit's place and the ten's place, *i* with the hundred's place and the thousand's place, and so on. The *varga* consonants are meant for *varga* places; the *avarga* for *avarga* places. For a number like 2500, denoted *mi*, one has to conceive the *varga* numeral *m* for 25 as being attached to the *varga* place 10^2 while for a number like 9000, denoted *si*, one sees the *avarga* digit *s* for 9 as being put in the *avarga* place 10^3.

Note the explicit use of a term for zero (*kha*) to denote a notational place – commentators like Bhāskara I and Sūryadeva have clarified that *kha* denotes *śūnya* and that *khadvinavake* refers to the eighteen [places] marked by zeros. Thus, unless a specific consonant-numeral occupies a certain place (10^x) in a number, by default, the place gets marked by a zero. We see here the occurrence of the concept of the mathematical zero as a place-marker in a place-value system.

Āryabhaṭa did not mention (or use) the alphabetical notation in the Gaṇita (mathematics) section. The role of this innovation was consciously restricted to the concise representation of large astronomical numbers in the Gītikā section. The notation was not suitable (and was not meant to be used) for performing arithmetic operations. Efficient systems, based on the decimal place value and zero, were already in vogue in India for that purpose. An intricate application of these principles can be seen in Āryabhaṭa's algorithms for finding square root and cube root (Gaṇita 4,5) a slight variant of which is now taught in school arithmetic. In the very second (and the first technical) verse of Gaṇita, Āryabhaṭa described the decimal system. Verbal decimal nomenclature (the number-vocabulary that we use in speech in Sanskrit-based vernaculars) dates back to the Vedic Saṁhitā composed in a period of remote antiquity. Expressions like *sapta śatāni viṁśatiḥ* (720), *sahasrāṇi śatā daśa* (1110) and *ṣaṣṭiṁ sahasrā navatiṁ nava* (60099) occur in the Ṛgveda (I.164.11, II.1.8, I.53.9); a verse of Medhātithi in the Vājasaneyī Saṁhitā (XVII.2) of the Śukla Yajurveda contains a list of single-word-names for powers of ten up to 10^{12} – each decuple term defined as ten times its predecessor.

The idea of using letters to denote numbers can be traced to the great ancient grammarian Pāṇini (c. 700 BCE vide [1]) who used vowels of the Sanskrit alphabet to signify numbers (*a*=1, *i*=2, *u*=3, etc.). Āryabhaṭa might have also been influenced by Sanskrit grammar and prosody where single consonants have sometimes been used to define objects to which frequent references have to be made. The trick of employing a base-100 scheme for codifying large numbers might have been inspired by the occasional use of the centesimal scale in Vedic literature (see [2], p 31, for examples). For instance, in the Taittirīya Upaniṣad (II.8), a centesimal scale is adopted to describe different orders of bliss; it is said that Brahmānanda (the bliss of Brahman) is 100^{10} times a unit of human bliss.

It appears that Āryabhaṭa's system was not used by subsequent astronomers (except during commentaries on his relevant verses). Some of the words became too complicated for pronunciation (as the readers would have noticed). Besides, the system does not provide sufficient variety for sustained versification. Later Indian astronomers either continued with the standard verbal decimal terminology (now prevalent) and another ancient decimal word-numeral system often called the "Bh- ūtasaṅkhyā" (whose roots too can be traced to Vedic literature) or adopted a Classical alphabetical decimal notation called "Kaṭapayādi". The originator of Kaṭa- payādi is not known; but it is presumed that Āryabhaṭa knew the Kaṭapayādi system – his commentator Śūrya- deva remarked that the letters *kāt* had been used by Āryabhaṭa (in Gītikā 2) to distinguish his method from the Kaṭapayādi.

In the Bhūtasaṅkhyā [*Bhūta*: occurrence, existing, consisting of; *Saṅkhyā*: number, sum, total], a digit *n* was denoted by the name of a well-known object or idea which usually occurs with frequency *n* (or has *n* components). For instance, *candra* (or any of the several synonyms for Moon) stood for one, *netra* (eyes) for two, *kāla* (Time: past-present-future) for three, *yuga* (satya-tretā-dvāpara-kali) for four and so on; synonyms for the sky (like *kha*, *śūnya, pūrṇa,* *randhra*) were used for zero. (For a list of such word-numerals used in Indian mathematical texts, see ([3], pp.332-3)). The word-numerals within a number were usually arranged in ascending powers of ten. The system was not concise – the number 4320000 would have a representation like "Śūnya-kha-pūrna-randhra-netra-kāla-yuga". However, the numerous choices for a digit, often with profound nuances, helped not only in maintaining the rhythms of the verses but also in infusing a poetic charm in the technical presentations. Due to this literary potential, word-numerals were sometimes used by stalwarts in astronomy even after the invention of compact alphabetical notations like Kaṭapayādi and Āryabhaṭa's scheme.

The Kaṭapayādi had four variants (see [1] for details and other notations) – we describe one of them. It is a form of decimal place-value notation using letters of the alphabet in place of numerical figures; the letters written in ascending powers of 10 (the digit in the unit's place written first followed by the digit in the ten's place to its *right*, and so on). The nine consonants from *k* to *jh*, as also the nine consonants from *ṭ* to *dh*, denoted (in usual order) the digits from 1 to 9; the five consonants from *p* to *m* denoted the digits 1 to 5 while the eight unclassed consonants from *y* to *h* denoted 1 to 8. Thus 1 could be denoted by any of the letters *ka, ṭa, pa, ya* (whence the name of the scheme). The consonants *ñ* and *n* denoted 0. All pure vowels not preceded by a consonant (i.e., occurring at the beginning) also denoted 0. A vowel joined to a consonant, or a consonant not joined to a vowel, did not carry numerical value. In a conjoined consonant, only the last one would denote a digit.

Inspite of being a verbal system, the Kaṭapayādi achieved the conciseness of a decimal place-value notation – any n-digit number got represented by an n-syllabic word. While it did not have the extreme brevity of Āryabhaṭa's scheme, it turned out to be more convenient than the latter for fulfilling metric requirements. For, although the Kaṭapayādi had the rigidity of position imposed by its place-value character, it actually attained greater flexibility from the near-absolute freedom in use of vowels and partial choices in use of consonants. The system provided considerable scope for skilled authors to compose sufficiently brief but pleasant-sounding chronograms often with connected meanings. For instance, the oft-used number 4320000 could be referred to by the 7-syllabic word *nānājñānapragalbha* (*nā*=*ñā*=*na*=0, *ra*=2, *ga*=3, *bha*=4) which is much less concise than Āryabhaṭa's 2-syllabic *khyughṛ* but more rhythmic and friendly; and certainly far more concise than a word-numeral representation. Terms were sometimes coined in such a manner that the Kaṭapayādi value of the chosen word would encode some numerical feature of the defined concept. An interesting example is the word *anantapura* – an Indian name for the lunar cycle. Apart from the literary nuance of the Moon's endless eastward (relative to the fixed stars) orbit in the boundless sky [*ananta*: endless, boundless, eternal, infinite, sky, atmosphere; *pura*: abode, from or towards the east], its Kaṭapayādi value 21600 (*a*=0, *na*=0, *ta*=6, *pu*=1, *ra*=2) is the number of minutes in a *pakṣa* (lunar half-month): $15 \times 24 \times 60$. It is said ([4], p.44) that the great philosopher Śankara was so named since the Kaṭapayādi value of the name (215; *śa*=5, *ka*=1, *ra*=2) indicated his birth-date – the fifth day of the first fortnight of the second month in the Indian lunar calendar.

The Kaṭapayādi scheme became popular in South India, especially in Kerala. It is applied in Karnatic (South Indian) music where the 72 *Janaka* (root) *rāgas* are classified into 12 groups of 6 rāgas each and numbered systematically, according to their notes, in such a way that the notes of a rāga can be quickly determined from the rāga number. The 72 rāgas are named in such a way that the Kaṭapayādi value of the first two syllables of its name gives the serial number of the rāga and hence its notes. For further discussion, see ([4]).

The Greeks too used the 27 letters of their alphabet to denote numbers – the first nine letters denoting numbers from 1 to 9, the next nine letters denoting the first nine multiples of 10 and the remaining nine letters denoting the first nine multiples of 100. A stroke or dot was used to indicate multiple of 1000 and the symbol M for multiplication by 10000.

The master-stroke of Āryabhaṭa's consonant-numeral system lies in the judicious application of the place-value idea – the successful imparting of high place-values to the vowels (which, in the Sanskrit script, tend to merge with the consonants) – to achieve an extreme compression in the *verbal* depictions of numbers (making it much shorter than even the decimal place-value notation). Considering the central importance attached to Āryabhaṭīya (c. 500 CE), and the use of the scheme in crucial verses at the very outset of this influential treatise, the understanding of Āryabhaṭa's coding in Gītikā 2 must have been indispensable for all serious astronomy scholars. Subsequent astronomer-mathematicians would have imbibed its subtleties and Āryabhaṭa's verse would thus have played a significant role not only in making the ideas embedded in the place-value principle take deeper roots in the Indian mind but also in pushing it towards an organised development of symbolic algebra. By the time of his brilliant successor Brahmagupta (628 CE), symbolism had become firmly established in Indian mathematics and algebra began to flourish as a distinct discipline. It is significant that Brahmagupta mentioned *varṇa* – which means "letters of the alphabet" as well as "colour" – as symbols for the unknowns. In later times, unknown variables were explicitly named by distinct colours like *kālaka* (black), *nīlaka* (blue), *pītaka* (yellow), *lohita* (red) and so on, and symbolically represented by the first letter of the respective colour-names – *kā*, *nī*, *pī*, *lo*, etc. The systematic use of letters of the alphabet to denote unknown variables was a great step for the rapid progress of mathematics. (See [5] for related observations.)

Thus, even though Āryabhaṭa's notation itself was not adopted, it is likely to have had a profound impact on the mathematical thought in Classical India.

Behind and before this analytical keenness, covering it as in a velvet sheath, was the other great mental peculiarity of the race poetic insight. Its religion, its philosophy, its history, its ethics, its politics were all inlaid in a flower-bed of poetic imagery the miracle of language which was called Sanskrit or "perfected", lending itself to expressing and manipulating them better than any other tongue. The aid of melodious numbers was invoked even to express the hard facts of mathematics.

– *Swami Vivekananda VI.157–8.*

Suggested Reading

[1] B Datta and A N Singh, *History of Hindu Mathematics, Part 1: Numeral Notation and Arithmetic*, Motilal Banarasidass, Lahore (1935-38); Asia Publishing House, Bombay, 1962.

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[3] B V Subbarayappa and K V Sarma (ed), *Indian Astronomy; A Source-Book*, Nehru Centre, Bombay, 1985.

[4] A V Raman, The Katapayadi Formula and the Modern Hashing Technique, *IEEE Annals of the History of Computing*, Vol.19, pp.49-52, 1997.

[5] A K Dutta, Mathematics in Ancient India, *Resonance*, Vol.7, No.4, pp.11-13; No.10, p.6, 2002.

[6] M Monier Williams, *Sanskrit-English Dictionary*, Clarendon Press, Oxford, 1899; reprinted Munshiram Manoharlal, New Delhi. 2002.