Lagrange and Classical Mechanics

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It is said, only partly in jest, that while the French regard Joseph Louis Lagrange as one of their greatest figures, the Italians claim Giuseppe Luigi Lagrangia for their own. Similarly, both mathematicians and physicists think of Lagrange as a towering figure in their disciplines. He belonged to an age of such eminent personalities – some others of comparable distinction being Euler, Gauss, Hamilton and much later Poincaré.

Lagrange's most important contributions to physics are in the area of (classical non-relativistic) mechanics. The fundamental physical concepts and laws were developed over the 17th century by Galileo followed by Newton – the principle of inertia, the clear understanding of acceleration and force, and the putting together – by Newton's genius – of the three laws of motion. Over the succeeding century and more, great advances were made by Euler, Lagrange and others in expressing the physics of Galileo and Newton in successively more powerful mathematical forms. In the process, while the infinitesimal calculus of Newton and Leibnitz was greatly developed as the perfect language of mechanics, the extension to the partial differential calculus took place, and between them Euler and Lagrange created the beautiful calculus of variations.

The developments in mechanics most intimately linked with Lagrange are the introduction of generalized coordinates q, the role of a single function of coordinates and velocities – the Lagrangian $L(q, \dot{q})$ – in determining all the equations of motion as it were at one stroke, and the principle of least action which reached its most convenient and flexible form in Hamilton's work. So one speaks of the Newtonian, Lagrangian and Hamiltonian forms of mechanics – always the same physical ideas but expressed with increasing mathematical sophistication and power. The Euler–Lagrange form of the equations of motion have the appearance

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} - \frac{\partial L(q,\dot{q})}{\partial q} = 0.$$

The importance of Lagrange's method can be appreciated in the following manner. Suppose one starts with the Newtonian equations of motion for a system of interacting particles written, say, in Cartesian position vectors, velocities and accelerations. If one wishes to re-express these equations using some other more convenient configuration space variables (such as spherical polar coordinates for each particle), the brute force method would be to express each Cartesian variable in terms of the new ones and their time derivatives, and then substitute in the equations of motion. But this is invariably very cumbersome. If the equations of motion are derivable from a single Lagrange function, a vastly more efficient and economical procedure is to simply express the Lagrangian in terms of the new variables, and then proceed as before. Quite generally the process of passing from one system of generalized coordinates to another is greatly simplified as indicated in this 'commutative diagram':

$$
\begin{array}{ccc}
L(q,\dot{q}) & \xrightarrow{\qquad\qquad} & \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}} - \dfrac{\partial L}{\partial q} = 0 \\
\Big\downarrow \text{①} & & \text{③} \Big\downarrow \\
L'(q',\dot{q}') & \xrightarrow[\text{②}]{\qquad\qquad} & \dfrac{d}{dt}\dfrac{\partial L'}{\partial \dot{q}'} - \dfrac{\partial L'}{\partial q'} = 0
\end{array}
$$

Step (1) and followed by Step (2) is much easier than the direct step (3)!

The Lagrange point of view gives such a unified and concise overall view of a physical system – its constitution, description and dynamics – that in fundamental physics one always refers to any specific theory or system by speaking of its Lagrangian. Thus one speaks of the Lagrangian for conservative multi particle systems, for the Maxwell electromagnetic field, for particle mechanics in special relativity, for general relativity, etc. It works for both discrete particle systems and for continuous fields. It also makes the discussion of continuous symmetries very direct and elegant: symmetries of the Lagrangian imply invariances of the equations of motion, and in turn laws of conservation via Noether's Theorem.

In his work on perturbation theory in mechanics, Lagrange introduced an expression later called the Lagrange Bracket. In this and in other ways, he anticipated many formal features of the later canonical formalism of mechanics due to Hamilton and Jacobi.

A few words should be added concerning Lagrange and quantum mechanics! As students know, quantum mechanics initially grew out of the canonical phase space form – the Hamiltonian form – of classical mechanics. The key classical notions of Poisson brackets and canonical transformations found their counterparts in commutation relations and unitary transformations. As we saw earlier, the Hamiltonian form of classical mechanics had grown out of the preceding Lagrangian form. In 1933 Dirac asked the profound question: does the Lagrangian have any *direct* role to play in the formulation of quantum mechanics? He did show that there were formal analogies between certain classical expressions involving the Lagrangian, and quantities describing unitary transformations in quantum mechanics. Some years later Feynman carried Dirac's idea to completion and created the Path Integral formulation of quantum mechanics in which the Lagrangian – more precisely its time integral (the action) – is the principal ingredient.

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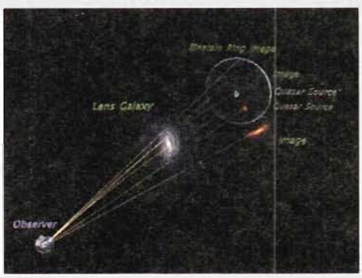

Front Cover:

'The illustration shows a dramatic consequence of the General Theory of Relativity, the bending of light by a gravitational field ('gravitational lensing'). Light, from a cosmologically distant quasar acting as the source, is intercepted by a galaxy (acting as the gravitational lens) along the line of sight to the observer (Earth). The bending of light can be strong enough to create multiple and distorted images, as in the case of the red quasar source and its two images (shown in red), and can even image the source into a ring of images called an Einstein Ring (blue ring in the illustration) if the source (shown also in blue) is well-aligned with the optic axis of the source-lens-observer system. The golden rays suggest the multiple paths light from the source actually takes to reach us in the presence of the gravitational lens, and the silver rays represent the directions from which we would observe the light to be coming (tangent to the actual light path at the observer).'