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Understanding Vanishing Energy During Charging of Capacitor Level

When a charged capacitor is connected to another uncharged capacitor, the charge on the first one is shared by both and some energy is lost through the connecting resistance. If this resistance is zero, it appears as though this energy is lost but cannot be explained mathematically. Actually, in such a case, it is necessary to consider circuit inductance which leads to oscillatory behaviour of the circuit.

There is a well known problem on two capacitors. It can be described as follows:

There are two capacitors of value C_1 and C_2. One of them is charged to voltage V. If this capacitor is connected to the other through a resistor R and a switch, and this switch is closed at t = 0, what will be the final voltage on the two capacitors and the final energy in the system?

After using a little calculus, one finds that the final voltage is $\frac{VC_1}{C_1+C_2}$ and energy is reduced from initial value of $\frac{V^2C_1}{2}$ to $\frac{V^2C_1^2}{2(C_1+C_2)}$. The remaining energy is lost as heat due to the current flowing in R. In a special case when $C_1 = C_2$, final voltage is $V/2$ and energy is reduced to half of initial energy.

Now, the next question is: *If the value of R is zero, what happens to the energy, as the final value of energy is independent of R?*

This problem is also seen in practical situations, where energy has to be transferred from one capacitor to another. For example, in pulse power systems, a Marx generator capacitor is first charged and its energy is transferred to a water line capacitor.

Keywords
Charging of capacitors, energy losses.

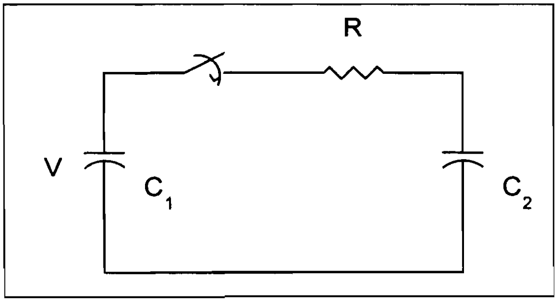


Figure1. C_1 is charged to V and connected to R and C_2 through the switch at t = 0.

Normally, in this problem, it is assumed that the circuit includes only two capacitors connected through a resistor. So, referring to the circuit in *Figure*1, we can write the current in the loop as

$$i(t) = \frac{V}{R} e^{\frac{-t}{RC}}, \text{ where } C = \frac{C_1 C_2}{C_1 + C_2}. \tag{1}$$

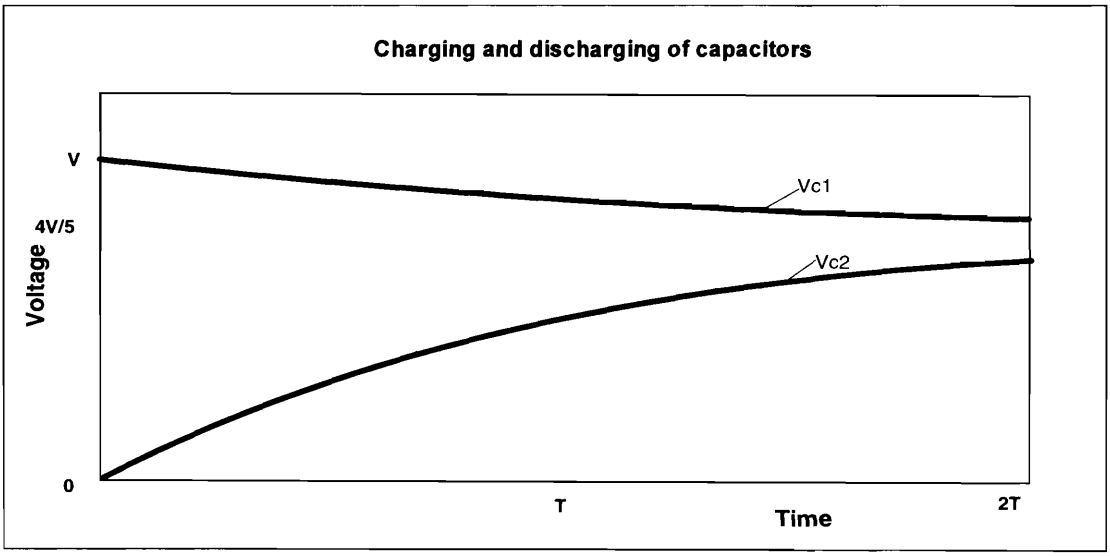
The voltages across capacitors C_1 and C_2 are found to be

$$VC_1 = \frac{VC_1}{C_1 + C_2} + \frac{VC_2}{C_1 + C_2} e^{\frac{-t}{RC}}, \tag{2}$$

$$VC_2 = \frac{VC_1}{C_1 + C_2} - \frac{VC_1}{C_1 + C_2} e^{\frac{-t}{RC}}. \tag{3}$$

A typical charging and discharging pattern for this analysis is shown in *Figure* 2, where $C_1 = 4C_2$. The final

Figure 2. Charging pattern of C_1 and C_2 when $C_1 = 4\,C_2$, Capacitors are connected through resistor; T = RC where $C = C_1C_2/(C_1 + C_2)$.



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voltage on capacitors is found to be

$$VC_1 = VC_2 = \frac{VC_1}{C_1 + C_2}. \quad (4)$$

The energy lost in the resistor, E_r, is then given as

$$E_r(t) = \int_0^t i^2 R dt$$

$$= \frac{V^2 C_1 C_2}{2(C_1 + C_2)}(1 - e^{\frac{-2t}{RC}}). \quad (5)$$

From this expression, it appears as though the energy lost is independent of R, since even if $R = 0$,

$$E_r(t) = \frac{V^2 C_1 C_2}{2(C_1 + C_2)}. \quad (6)$$

Therefore this approach leads to an ambiguous situation where even if $R = 0$, the energy still is lost from the circuit.

This problem is briefly explained at some places in the literature [1], and the explanation is that the energy is lost in heat and sparking in resistive circuits and electromagnetic radiation in underdamped circuits. But this answer is not entirely satisfactory, as it does not address the situation when $R = 0$.

In the derivation shown above, the assumption is that the inductance (L) is zero. As we know from basics of electrical engineering, the resistance arises due to opposition to the flow of charge carriers in the conducting medium, whereas the inductance is due to opposition to the rise of current. If we have two independent capacitors with one of them charged to a certain voltage and we wish to transfer the charge to second one, then this can happen only by flow of current. So even if we have made the circuit resistance zero somehow or the

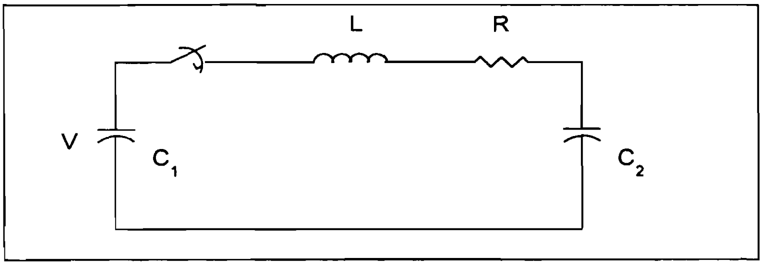

Figure 3. Circuit in Figure1 modified to include the effect of L.

other, it is not correct to assume that inductance also would be zero at the same time. Alternately this means that $L \leq 0$. Therefore, a better way to tackle the same problem is by including both R and L in the circuit.

Let us consider the circuit in *Figure* 3, which includes a series inductance L in addition to resistance R. Let the capacitor C_1 be charged to V, as before and at $t = 0$, switch is closed. So we can write,

$$L\frac{di}{dt} + Ri + \frac{1}{C1}\int idt + \frac{1}{C_2}\int idt = V \qquad (7)$$

This equation can be solved to get the value of current as

$$i(t) = \frac{V}{\omega L}e^{-at}\sin(\omega t), \text{for} \frac{L}{C} > \frac{R^2}{4}. \qquad (8)$$

where

$$a = \frac{R}{2L}, \; C = \frac{C_1C_2}{C_1 + C_2} \; \text{ and } \; \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}.$$

From this value of current, we can calculate the voltages across C_1, C_2 and L. It is obvious that this is a case of underdamped response, which results in oscillatory waveform which is continuously decreasing in amplitude due to the presence of R.

Now if we put $R = 0$ and $L \neq 0$, we get current as,

$$i(t) = \frac{V}{\omega L}\sin(\omega t) = CV\omega\sin(\omega t), \qquad (9)$$

since $a = 0$ and $\omega = \sqrt{\frac{1}{LC}}$.

The energy in the circuit is not lost, but keeps recirculating.

For this case, the values of voltages across capacitors can be found as,

$$VC_1(t) = \frac{C_2 V}{C_1 + C_2}\cos(\omega t) + \frac{C_1 V}{C_1 + C_2}, \tag{10}$$

$$VC_2(t) = \frac{C_1 V}{C_1 + C_2} - \frac{C_1 V}{C_1 + C_2}\cos(\omega t). \tag{11}$$

Thus, the total energy in the system can be calculated as sum of energies in each element, i.e.,

$$E_{\text{total}}(t) = E_{C_1}(t) + EC_2(t) + EL(t),$$

$$= \frac{1}{2}C_1[VC_1(t)]^2 + \frac{1}{2}C_2[VC_2(t)]^2 + \frac{1}{2}L[i(t)]^2 \tag{12}$$

After substituting values in equation (12) and simplifying, we get $E_{total}(t) = \frac{1}{2}C_1V^2$, which is the original energy.

This shows that the energy in the circuit is not lost, but keeps recirculating when $R = 0$ but $L \leq 0$. Now if we get a situation where $L \to 0$, oscillating frequency tends to infinity. At the same time, the average value of voltage across C_1 and C_2 is $\frac{C_1V}{C_1+C_2}$. This means that the final voltage seen on each capacitor is same as the one that was found in our first derivation based on R only. The remaining energy is circulating in the circuit. The voltage waveforms across capacitors for a typical case with $C_1 = 4C_2$ is shown in *Figure* 4. It can be seen that the oscillations are taking place around dc voltage of $4V/5$.

Thus when one charged capacitor is connected to another one, the mode of transfer of energy, (i.e. exponential or oscillatory) depends on the value of R, C and L in the circuit. The energy is lost in the resistor and this loss can be by heating or by radiation depending on the value of L and C. However, there is no loss of energy from the circuit in a situation where $R = 0$.

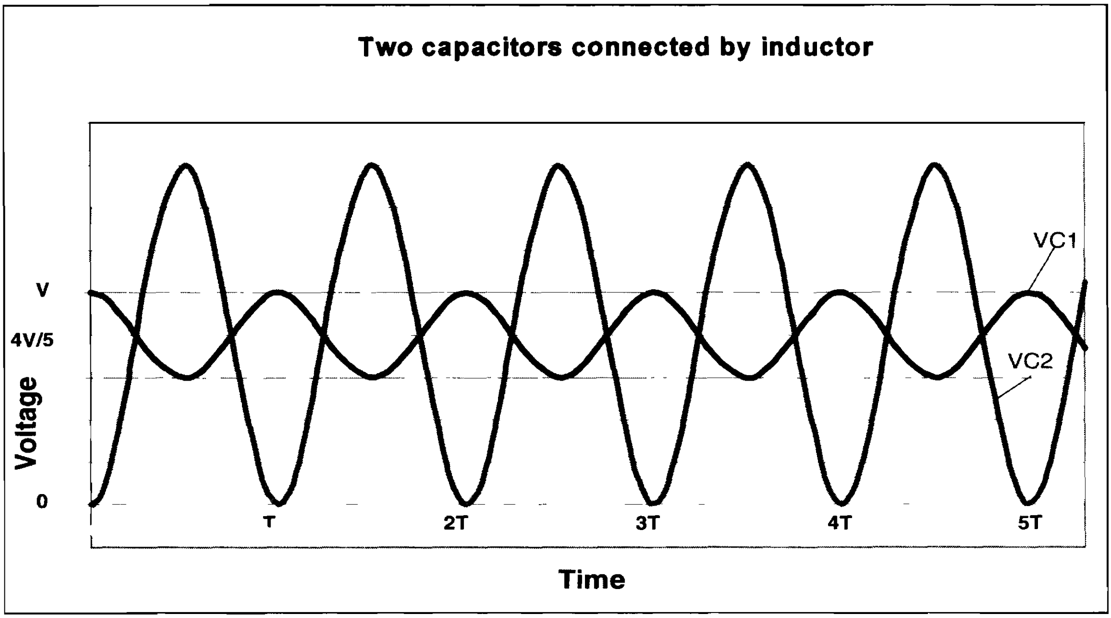

Figure 4. Charging pattern of C_1 and C_2 where $C_1 = 4\,C_2$, capacitors connected through inductor, $T = 2\pi(LC)^{1/2}$, where $C = C_1C_2/(C_1 + C_2)$.

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Suggested Reading

[1] E Hughes, *Electrical Technology*, Longman Group Ltd., 1969.

Nothing tends so much to the advancement of knowledge as the application of a new instrument. The native intellectual powers of men in different times are not so much the causes of the different success of their labours, as the peculiar nature of the means and artificial resources in their possession.

Sir Humphrey Davy