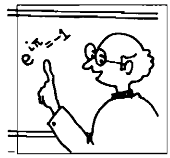

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Two-Dimensional Collision – at Higher Secondary Level

The treatment of two-dimensional collision of bodies in textbooks at various levels was examined. It is found that while many books do not mention impact parameter at all, some books show it in a diagram, though they do not include it in the mathematical treatment that follows. In this article, the importance of impact parameter is highlighted and a simple treatment which does not go beyond the higher secondary level is developed. A simple experiment is suggested.

1. Prologue

Two-dimensional, or angular, collision of masses is a topic for higher secondary (HS) and undergraduate (UG) physics beginners. We were recently involved in producing some course material on physics for in-service training of HS teachers. In the course of this development, we found that there are certain misconceptions in the minds of teachers and authors of books in this regard. We suggest an improved, yet simple, treatment of the topic at UG or even the HS level. This is possible by restricting

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restricting to hard-sphere scattering – that is, by assuming that two bodies scatter only when they touch each other. This includes the collision of small classical bodies like balls, stones and vehicles on the earth. It excludes long-range potential scattering.

We have found that textbooks, Indian or foreign, dealing with 2-D collision can be divided into three categories: (1) Books [1-6] which do not give any idea of the impact parameter; (2) Books [7-9] which show the real-space diagram, show the impact parameter, but do not take it into account in further analysis; (3) Books [10-12] which use impact parameter in further analysis of scattering in a long-range potential. Books of category 1 give incomplete and wrong picture, those of category 2 give the correct but incomplete picture, while books of category 3 take us to higher level.

2. The Present Treatment

Consider that two masses, m_1 and m_2, moving with initial velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, collide with each other, and move away in different directions with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. It reduces to a head-on collision if and only if the two centres of mass initially move along the same direction and along the same line. The case of head-on collision has been discussed earlier [13], and a simple, extremely low-cost, classroom demonstration as well as experiment suggested for it.

In the usual textbook treatment of two-dimensional collision, a special case is considered for simplifying the algebra, in which the target particle, Particle 2, is taken at rest, so that its velocity $\mathbf{u}_2 = \mathbf{0}$. The direction of motion of the incident particle (also known as the *projectile*), which is the direction of its velocity $\mathbf{u}_1$, is called the *forward direction.* After collision, the particles travel in different directions, the incident Particle 1 in a direction making an angle θ_1 and the target Particle 2 making an angle θ_2, with the forward direction. A diagram such

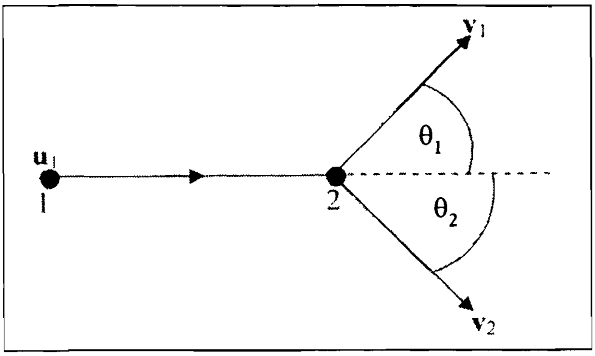

Figure 1. A diagram for 2-D collision shown in some books. If it is a real-space diagram, the line of travel of Mass 1 should not pass through the CM of Mass 2. If it is momentum-space diagram, the bodies should not be shown.

as that shown here in *Figure* 1 is given. Equations are written down for conservation of energy and momentum, with the difference that here momentum will have two components, parallel and normal to the forward direction. Thus we get the following three equations:

$$m_1 u_1^2 = m_1 v_1^2 + m_2 v_2^2, \tag{1a}$$

$$m_1 u_1 = m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2, \tag{1b}$$

$$m_1 v_1 \sin\theta_1 = m_2 v_2 \sin\theta_2. \tag{1c}$$

In this exercise, there are four unknowns, the magnitudes v_1, v_2, and the directions θ_1, θ_2, of the final velocities, though we have only three equations above. Some general relations are derived, by eliminating one or the other of these variables, and then some special cases are treated. Let us define the mass ratio of the incident particle to the target particle as $\alpha = m_1/m_2$ Then one can eliminate the initial velocities and get equations relating the two angles, followed by one equation relating all the four parameters, in the form

$$\tan\theta_1 = \frac{\sin 2\theta_2}{\alpha - \cos 2\theta_2}, \tag{2a}$$

$$v_1 = \frac{v_2 \sin\theta_2}{\alpha \sin\theta_1}. \tag{2b}$$

Alternatively, one could get v_1 and v_2 in terms of u_1 and the angles. But with all this, one is left with the

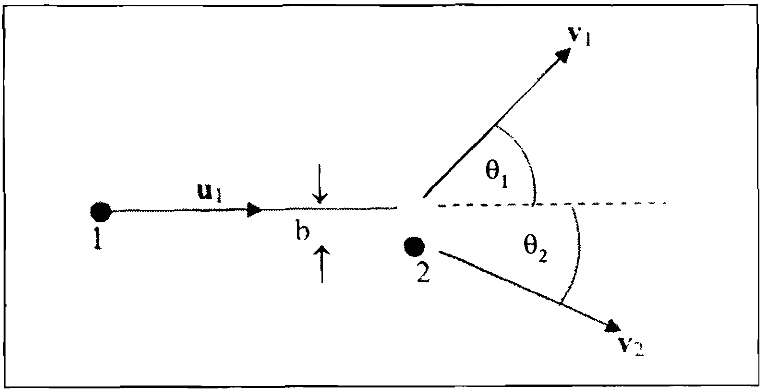

Figure 2. The correct real-space diagram for 2-D collision.

impression that we have only three equations and four unknowns, and so we cannot obtain unique solutions.

3. What is Missing?

It must be understood that *Figure* 1 is a mixture of real space and momentum space pictures. So far as momentum is concerned, there is only one initial momentum vector, but two final momentum vectors. In real space, it would just be a head-on collision if the line of travel of the centre of mass (CM) of the incident Particle 1 passes through the CM of the target Particle 2. In order to have a 2-D collision, the CM of Particle 2 must be off the line of travel of CM of Particle 1. This necessitates introducing an additional parameter.

This additional parameter which makes all the final variables unique is the perpendicular distance, or the shortest distance, of the target particle from the forward direction. It is known as the *impact parameter*. Let us denote it by b. This is shown in *Figure* 2. It is now obvious that collision will take place, under the assumption of hard-sphere scattering, only if the two bodies are real bodies with finite sizes, and not just point particles. One lesson to learn from this is that there cannot be 2-D collision in the case of mathematically point particles, where either it will be a head-on collision or there would be no collision.

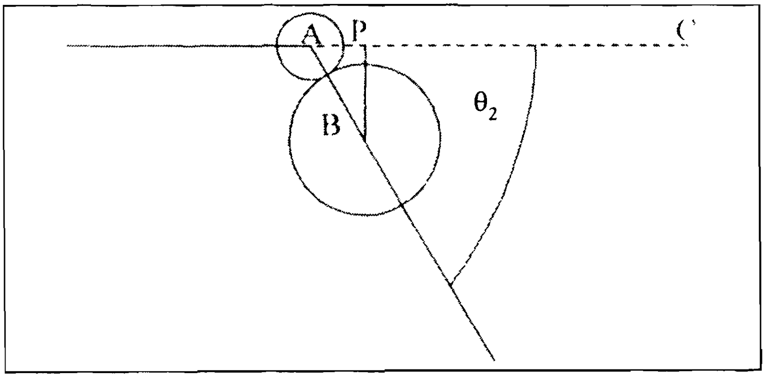

Figure 3. The moment of collision in real space when two smooth spheres touch each other.

Note that the assumption of hard-sphere scattering allows us to confine our treatment to HS level, which is the purpose of this article. To simplify the situation further, we consider both bodies to be spherical in shape, of radii R_1 and R_2, respectively, with smooth surfaces, so that there is no friction and no resulting rotation after collision. Collision will occur when the distance between the two centres, which is also the distance between their CMs, is exactly equal to $R_1 + R_2$.

4. The Impact-Parameter Treatment

Refer to *Figure* 2, with the target particle at rest, $\mathbf{u}_2 = \mathbf{0}$. The energy and momentum equations remain the same as equations(1). In the simple case described above, smooth spheres with hard-sphere scattering, the angle θ_2 at which Particle 2 is scattered is determined by the configuration geometry at the moment of collision – Particle 2 will fly off along the line joining the two centres of spheres, in the direction from Particle 1 to Particle 2. *Figure* 3 shows the moment of collision. Let A and B be the centres of the two spheres at this moment. Let AC be the forward direction. Draw a perpendicular BP from the target to the forward direction. Then we have BP = b, and AB = $R_1 + R_2$. Particle 2 will fly off along the line AB, making an angle θ_2 with AC. Then it is clear that

$$\sin\theta_2 = \frac{b}{R_1 + R_2} \equiv p, \qquad (3)$$

which defines the parameter p as the fractional ratio of the impact parameter and the sum of radii. It could be called the *fractional impact parameter*. It should be clear that p lies between 0 and 1. Moreover $p = 0$ corresponds to head-on collision. For $p \geq 1$, there is no physical contact and hence no collision. (One must remember that for long-range potentials, scattering can take place for any value of b.)

Equation (3) is the missing equation, which is generally not mentioned in books. But it is extremely important, because there can be no 2-D collision unless the impact parameter is nonzero. Once again we emphasize that there is a difference between a real-space diagram and a momentum-space diagram. Once we have equation (3), the rest becomes easy, and we can obtain the other three final quantities uniquely. Here we show only some mathematical steps and the final results. We have

$$\sin 2\theta_2 = 2p(1-p^2)^{1/2} \quad \cos 2\theta_2 = 1 - 2p^2 \tag{4a}$$

$$\tan\theta_1 = \frac{2p(1-p^2)^{1/2}}{\alpha - 1 + 2p^2}, \sin\theta_1 = \frac{2p\,(1-p^2)^{1/2}}{[(\alpha-1)^2 + 4\alpha p^2]^{1/2}}. \tag{4b}$$

$$v_2 = \frac{2\alpha(1-p^2)^{1/2}}{1+\alpha} u_1. \tag{4c}$$

$$v_1 = \frac{v_2 \sin\theta_2}{\alpha \sin\theta_1} = \frac{[(\alpha-1)^2 + 4\alpha p^2]^{1/2}}{1+\alpha} u_1. \tag{4d}$$

All the four quantities are determined in terms of the two parameters of the system, the mass ratio and the fractional impact parameter p, and initial velocity u_1 of the incident particle.

We have noticed wrong statements in certain printed materials in regard to the direction of travel of the target particle. Writing equation (2b) in the form

$$v_2 \sin\theta_2 = \alpha v_1 \sin\theta_1, \tag{5}$$

we see that, as $\theta_1 \to 0, \theta_2$ also tends to zero. Some authors then suggest that for a heavy projectile and light target, both particles move away close to the forward direction. But this is not right. It is one thing to say that, in equation (5), if $\theta_1 = 0$ then $\theta_2 = 0$, which is of course right, and simply corresponds to head-on collision. But it is important to understand that $\theta_1 = 0$ is *quite different from* $\theta_1 \to 0$ It is clear from equation (5) that if $\sin\theta_1$ is small but α is large, θ_2 could have any value. As discussed above, θ_2 is entirely decided by the geometry at the moment of collision and, in a sense, is under our control.

5. Special Cases

Now we can consider special cases by varying α and p. The very first thing that can be and should be verified is that the phenomenon reduces to head-on collision for $p = 0$.

(a) Head-on collision, $b = 0$, $p = 0$:

Equations (4) readily show that in this case, we get

$$\theta_1 = \theta_2 = 0, \quad v_1 = \frac{\alpha - 1}{\alpha + 1} u_1, \quad v_2 = \frac{2\alpha}{\alpha + 1} u_1. \qquad (6)$$

Thus we get back the formulas for head-on collision. Various sub-cases of this for different α have been discussed earlier [13].

(b) The case of $p = 1/\sqrt{2}$:

To get a feel of the role played by the impact parameter, we consider the case when

$$b = (R_1 + R_2)/\sqrt{2}, \quad p = 1/\sqrt{2}. \qquad (7)$$

In this case, triangle APB becomes an isosceles triangle, with AP = PB, and equation (4a) gives $\theta_2 = \pi/4$. The other final parameters come out to be

$$\sin\theta_1 = \frac{1}{[(\alpha - 1)^2 + 2\alpha]^{1/2}}, \qquad (8a)$$

When a sphere collides with another equal-mass stationary sphere at any angle (any impact parameter), the two go off at right angles to each other.

$$v_1 = \frac{[(\alpha - 1)^2 + 2\alpha\,]^{1/2}}{1+\alpha} u_1, \tag{8b}$$

$$v_2 = \frac{\sqrt{2}\alpha}{1+\alpha} u_1. \tag{8c}$$

Once again, we can consider cases for different α For equal masses, α =1, the final parameters come out to be

$$\theta_1 = \theta_2 = \pi/4, \;\; v_1 = v_2 = u_1/\sqrt{2}, \tag{9}$$

which has a simple explanation in terms of what is happening. When a sphere collides with another stationary sphere of equal mass in such a manner that the line joining their centres makes an angle of 45^o with the forward direction, irrespective of their radii, both spheres scatter at an angle of 45^o on either side of the forward direction, and go off with the same speed $u_1/\sqrt{2}$ One can verify that both energy and momentum are conserved with these values.

(c) Equal masses, α =1, any p:

Now we consider two equal-mass spheres, for any impact parameter. For this case, equation (4) reduces to

$$v_1 = pu_1, v_2 = (1-p^2)^{1/2}u_1, \sin\theta_2 = p\sin\theta_1 = (1-p^2)^{1/2} \tag{10}$$

It is obvious from here that $\theta_1 + \theta_2 = \pi_2$, which is a well-known result. When a sphere collides with another equal-mass stationary sphere at any angle (any impact parameter), the two go off at right angles to each other.

As sub-cases of this case, let us see what happens if p is small ($p \to 0$) and if p is large ($p \to 0$). For small p, the target particle scatters at a low angle, though it takes up the major fraction of the energy of the incident particle, while the incident particle scatters through a large angle and moves away with a small velocity. For the other case, $p \to 1$, the situation is exactly the opposite.

6. Impact Parameter in the General Case

It was easy to define the impact parameter in the case when the target particle is at rest. But if $\mathbf{u}_2 \neq 0$, how do we define the impact parameter?

We go to the target frame of reference, in which the target particle is at rest, as if we are sitting on it, and the incident particle approaches it with a relative velocity $\mathbf{u}_1 - \mathbf{u}_2$. This allows us to define the impact parameter as before. We calculate the final parameters as above, in the target frame of reference, and denote them by $\mathbf{v}'_1$, $\mathbf{v}'_2$, θ'_1, θ'_2. Then in the laboratory frame the new velocities are simply obtained by adding $\mathbf{u}_2$ to the target-frame velocities (remembering that in the lab frame, the target was moving with a constant velocity $\mathbf{u}_2$). The lab-frame scattering angles, θ_1, θ_2 can be obtained by referring to *Figure* 4. Here $\mathbf{v}_1$ is the target-frame velocity of the incident particle after scattering, and θ'_1 is the angle at which it scatters from the forward direction. The lab-frame velocities after scattering are given by

$$\mathbf{v}_1 = \mathbf{v}'_1 + \mathbf{u}_2, \mathbf{v}_2 = \mathbf{v}'_2 + \mathbf{u}_2. \qquad (11)$$

The forward direction in this case is the direction of $\mathbf{u}_1 - \mathbf{u}_2$. The angles made by $\mathbf{v}_1$ and $\mathbf{v}_2$ with this forward

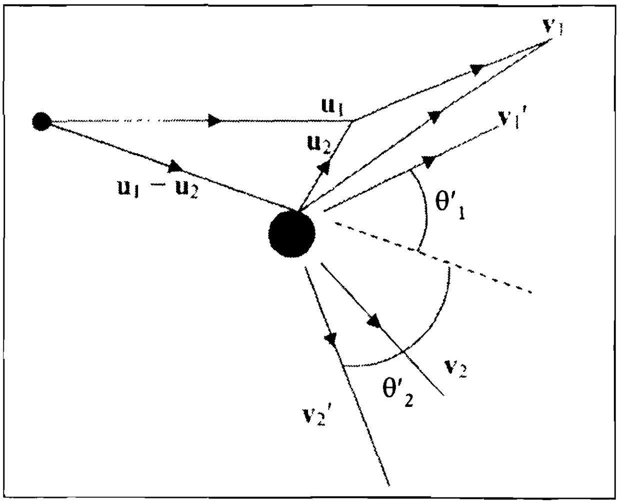

Figure 4. Relating the target frame parameters with the laboratory frame parameters.

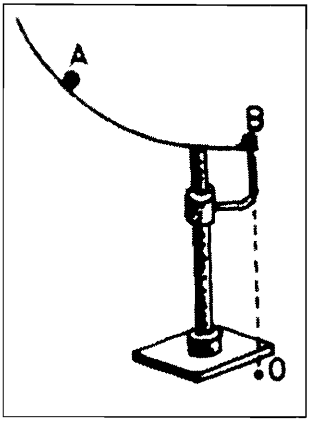

Figure 5. A simple low-cost experiment for 2-D angular collision of balls.

direction are θ_1, θ_2, which can be easily obtained from the geometry of *Figure* 4.

7. A Low-Cost Experiment

The Pune team has built a channel (see *Figure* 5) which allows a ball to fall from a given initial height, and travels along a horizontal path when it leaves the channel. Another ball is placed in its path on a rotatable sleeve so as to vary the angular placement of this ball. In this experiment, Ball A has pre-determined initial energy and momentum, while Ball B is stationary. The balls are allowed to fall on a tray and their positions are marked. As in the experiment on head-on collision [13], here too the measurement of velocities is converted into measurement of distances. Point O in *Figure* 5 is the projection on the tray of the point where the ball(s) leaves the channel. The various momenta can be determined from the horizontal distance the balls have traveled, and their direction. This allows one to get a feel of the initial energies and momenta, as well as the final parameters. One can try to draw the parallelogram of momenta. The experiment can be performed with different balls, such as of steel, glass, wood, plastic, cork, etc, with different mass ratios. One also gets an idea of the loss of energy in the collision. One sees that no collision is ever perfectly elastic, and the loss depends on the nature of the balls. This experiment was taken from PSSC Physics and has been described in *A World-View of Physics* [14].

8. Summary and Simple Applications

In this article, we have emphasized the importance of impact parameter in 2-D collisions. We have shown that the treatment can still be kept at HS level under the assumption of hard-sphere scattering of smooth spheres. The second point that is brought out is that, under this assumption, the scattering angle of the target is decided by the fractional impact parameter.

One can think of everyday examples of 2-D collision in simple situations. The simplest such case is the game of carrom, in which a player hits a circular coin with a larger and heavier circular striker. The striker is the incident particle and the target is stationary, and hence we have a fairly close application, though friction cannot be neglected in the real case. In order to pocket the coin, the player has to make it move along a certain direction. In our context, this means that the desired angle θ_2 is known beforehand. A good player adjusts the line of travel of the striker, and hence the impact parameter, to achieve the desired objective. A somewhat more complicated case occurs in the game of billiards, the complication arising due to the spherical balls and their resulting rotation, though there is one simplifying factor here – the striker and the target have the same mass.

Suggested Reading

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