The Einstein–Podolsky–Rosen Paper

An Important Event in the History of Quantum Mechanics

S Chaturvedi, N Mukunda and R Simon

(left) S Chaturvedi is Professor at the School of Physics, University of Hyderabad since 1993. His current research interests are mathematical physics, geometric phases, coherent states, Wigner distribution, and quantum entanglement.

(right) N Mukunda is at the Centre for High Energy Physics, IISc, Bangalore. His interests are classical and quantum mechanics, theoretical optics and mathematical physics.

(center) R Simon is with the Institute of Mathematical Sciences, Chennai. His research interests are in quantum information science and quantum optics.

Keywords
Quantum mechanics, EPR argument, local realism.

1. Introduction

In 1935, Albert Einstein, Boris Podolsky and Nathan Rosen published a paper titled 'Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?' in the *Physical Review*, Volume 47, pages 777-780. Generally known as the EPR paper, it has had an enormous influence, despite its short length, on the interpretation and further analysis of quantum mechanics. It immediately led to a reply from Niels Bohr in a paper with exactly the same title (except that 'Be' was replaced by 'be') in the same year, on pages 696-702 of the next volume of the same journal.

The aim of this article is to describe the background to and content of the EPR paper, viewed critically and keeping in mind a reader who begins her exploration of this subject with a study of this famous paper. We cover briefly: the discovery of quantum mechanics and the development of its traditional interpretation; Einstein's unhappiness with this interpretation; Bohr's Complementarity Principle expressed in a symbolic manner; a resumé of Einstein's general viewpoints on the notions of separability and objective reality; the structure and main arguments of the EPR paper; Bohr's point of view and rejoinder; Bell's analysis of locality and realism; and an instructive example due to Hardy bringing out sharply the difficulties with realism in quantum theory. We conclude with a brief summary and the lessons to be learnt from this famous episode in physics.

2. Discovery of Quantum Mechanics and Development of its Interpretation

Quantum mechanics was discovered in three major steps over the space of less than a year during 1925-26: Heisenberg's matrix mechanics in the summer of 1925, Dirac's more symbolic quantum mechanics in autumn-winter 1925, and Schrödinger's wave mechanics in winter-spring 1926. In the work of Heisenberg and of Dirac, the emphasis was on the description of observables or dynamical variables for quantum systems, especially the noncommutative nature of their multiplication. In Schrödinger's work the concept of a general state and its description by means of a wave function was emphasized, of course along with an equation of motion determining its evolution in time. This latter was the quantum mechanical replacement for Newton's equation of motion in classical mechanics. In Dirac's hands, the wavefunction concept led to the formulation of the fundamental Superposition Principle of quantum mechanics. In June 1926, Max Born proposed the interpretation of the wave function in terms of probabilities, something at variance with Schrödinger's own initial expectations.

Quantum mechanics was discovered in three major steps over the space of less than a year during 1925-26.

The traditional interpretation of quantum mechanics emerged during 1926 and early 1927, involving intensive discussions among Bohr, Heisenberg and Pauli. Both Heisenberg's Uncertainty Principle and Bohr's Complementarity Principle resulted from these discussions. In September 1927, at a conference in Como to observe the centenary of the death of Alessandro Volta, Bohr made the first public presentation of his Complementarity Principle, but failed to communicate his ideas effectively. Einstein was not present at this conference.

Einstein was unhappy with the traditional interpretation of quantum mechanics, and till the end of his life he was unwilling to accept it. His initial attitude was that quantum mechanics was incorrect and internally

Einstein believed that there exist situations in Nature which could not be described in the framework of quantum mechanics.

inconsistent. He believed he could create experimental situations where the Uncertainty Principles would be violated. He made two important attempts to do so. The first was at the 5th Solvay Congress in October 1927, where he proposed thought experiments to show how the position-momentum uncertainty principle

$$\Delta q\, \Delta p \;\geq \hbar/2 \tag{1}$$

could be 'beaten'. The second was three years later, at the 6th Solvay Congress in October 1930; this time his proposed thought experiment was to disprove the energy-time uncertainty relation

$$\Delta t\, \Delta E \gtrsim \hbar, \tag{2}$$

known as the *Bohr Uncertainty Principle*. On both occasions, Bohr was able to pinpoint the errors in the argument and thus rescue the interpretation and consistency of quantum mechanics.

After these episodes Einstein altered his stand: he conceded that quantum mechanics was internally consistent but claimed that it was incomplete. He believed that there exist situations in Nature which could not be described in the framework of quantum mechanics. It was this train of thought that ultimately led to the 1935 EPR paper.

3. The Standard Interpretation

At this point it is useful to recapitulate very briefly the standard interpretation of quantum mechanics. Let S denote some physical system. Its quantum mechanical description involves two sets of mathematical quantities, with associated physical meanings:

Physical quantities or observables or dynamical variables:

(a) represented (generally) by noncommuting hermitian operators $\hat{A}, \hat{B}, \ldots$;

(b) (pure) States describable by vectors $|\psi\rangle$, $|\varphi\rangle$, in a Hilbert space H, subject to the Superposition Principle.

$$(3)$$

The operators $\hat{A}, \hat{B}$, act on the Hilbert space $\mathcal{H}$ characteristic of the system. Given that S is in the state $|\psi\rangle$, suppose an experiment is set up to measure $\hat{A}$. We can ask: In principle, what can the results of the measurement be? Knowing that the state is $|\psi\rangle$, what will the results be, and what is the probability for each possible result?

To answer these questions, we need to study the eigenvalues and eigenvectors of $\hat{A}$. Assume for simplicity that the eigenvalues are discrete and nondegenerate. Denote them and the corresponding eigenvectors by a_j and $|a_j\rangle$ respectively. Then we have

$$\begin{aligned} \hat{A}|a_j\rangle &= a_j\,|a_j\rangle, \quad a_j \text{ real}, \\ \langle a_j|a_k\rangle &= \delta_{jk} \end{aligned} \tag{4}$$

and $\{|a_j\rangle\}$ forms an orthonormal basis for $\mathcal{H}$ (reality of the eigenvalues and orthogonality and completeness of the eigenvectors are assured by the hermiticity of $\hat{A}$). Then expand $|\psi\rangle$ in this basis:

$$\begin{aligned} |\psi\rangle &= \sum_j \psi_j|a_j\rangle, \quad \psi_j = \langle a_j|\psi\rangle, \\ \langle\psi|\psi\rangle &= \sum_j |\psi_j|^2 = 1. \end{aligned} \tag{5}$$

We assumed here that $|\psi\rangle$ is normalised to unit length. The answers to our questions are: the possible results of measurement of $\hat{A}$ are the eigenvalues a_j. In the (pure) state $|\psi\rangle$, the probability of obtaining the result a_j is $|\psi_j|^2$. To this is added the collapse postulate: If the result a_j is obtained, then $|\psi\rangle$ collapses to (is to be replaced by) $|a_j\rangle$, which is to be used for discussing further observations. Clearly, further repeated measurements of $\hat{A}$, on the collapsed state $|a_j\rangle$, will result in the same eigenvalue a_j. The experimental verification of the predicted probabilities $|\psi_j|^2$ therefore requires making measurements on a large collection of identically prepared copies of the state $|\psi\rangle$. For the time dependent state vector $|\psi(t)\rangle$, the equation of motion is the time-dependent Schrödinger equation. This is to be used only in between measurements.

For a composite system $S = A + B$ made up of subsystems A and B, the state space is the tensor product of the individual spaces:

$$\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{6}$$

We will come back to this later.

4. The Complementarity Principle

Bohr's Complementarity Principle cannot be easily formulated with the same precision with which Heisenberg's Uncertainty Principle is formulated. We give here a schematic account of Bohr's Complementarity Principle. Every announcement of an experimental result 'R' must be accompanied by a statement of the experimental set up 'E' that led to it, so we must always speak of and deal with *pairs* (E, R). In turn, E is a combination of the system S (in some state $|\psi\rangle$), and apparatus $\mathcal{A}$ or $\mathcal{B}$ or ... constructed or designed to measure an observable $\hat{A}$ or $\hat{B}$ or ..... $\mathcal{A}$ itself symbolically represents the apparatus and the actual carrying out of the experiment. So we ultimately deal with triples $(S, \mathcal{A}, R)$, where $\mathcal{A}$ specifies $\hat{A}$, and the state $|\psi\rangle$ is left implicit.

In the classical case we say: we can delete $\mathcal{A}$ from the triple $(S, \mathcal{A}, R)$, and we can claim and imagine that the system S possessed the value R for the physical quantity $\hat{A}$ at the time of the measurement. We say that this is so, independent of the apparatus $\mathcal{A}$ and the act of measurement. So we can simply deal with pairs (S, R), the relevant values of time being left implicit.

In quantum mechanics, on the other hand, we cannot do so, we cannot remove $\mathcal{A}$ from the description. In the language of Bohr, the entire triple $(S, \mathcal{A}, R)$ is a 'phenomenon' not reducible to anything more elementary. So we have the situation:

$$(S, \mathcal{A}, R) \begin{cases} \text{Classical: delete } \mathcal{A}, \text{ keep } (S, R), \text{ say } S \\ \qquad\qquad \text{has value } R \text{ for } \hat{A}. \\ \text{Quantum: cannot delete } \mathcal{A}, \\ \qquad\qquad \text{retain triplet as a whole.} \end{cases} \tag{7}$$

Thus: if the experimental arrangements $\mathcal{A}_1, \mathcal{A}_2$ to measure $\hat{A}_1, \hat{A}_2$ are mutually exclusive (for example, Stern–Gerlach apparatuses to measure the spin components S_x, S_y of a spin $\frac{1}{2}$ object in two different directions), it means that $\hat{A}_1$ and $\hat{A}_2$ do not commute and cannot be measured simultaneously. We can have either $(S, \mathcal{A}_1, R_1)$ or $(S, \mathcal{A}_2, R_2)$ at a given time, one phenomenon or the other. We cannot think of S possessing values R_1 and R_2 for $\hat{A}_1$ and $\hat{A}_2$ simultaneously.

If the experimental arrangements $\mathcal{A}_1, \mathcal{A}_2$ to measure $\hat{A}_1, \hat{A}_2$ are mutually exclusive, it means that $\hat{A}_1$ and $\hat{A}_2$ do not commute and cannot be measured simultaneously.

For Bohr, S was quantum and $\mathcal{A}$ classical, and he said the whole $S + \mathcal{A}$ was 'unanalysable' In von Neumann's treatment, $S, \mathcal{A}$ and $S + \mathcal{A}$ are all quantum systems, so $S + \mathcal{A}$ is subject to a Schrödinger equation. But then the collapse rule cannot be derived from the Schrodinger equation, at least not in any simple manner.

5. Einstein's General Viewpoints

It was mentioned earlier that Einstein never accepted the standard interpretation of quantum mechanics. He also did not agree with Bohr's Complementarity Principle. About this he said:

"Of the 'orthodox' quantum theoreticians whose position I know, Niels Bohr's seems to me to come nearest to doing justice to the (EPR) problem .... Bohr's principle of complementarity, the sharp formulation of which I have been unable to achieve despite much effort I have expended on it"

'Objective reality': physical systems possess numerical values for their properties independent of our observations of them.

In contrast to the Complementarity Principle, he believed in 'objective reality': physical systems possess numerical values for their properties independent of our observations of them. He also insisted upon some other important ideas, even if they were not always precisely expressed. Here are two of them:

Separability: This was a necessary ingredient of any theory in physics. Two systems S_1 and S_2 which are spatially far away from each other must be 'independent' they cannot 'influence' one another. There cannot be any action at a distance.

This requirement is meaningful even nonrelativistically, special relativistic locality is more precise and refined.

Real state of a system: this is indicated only qualitatively: it is not something

"... immediately accessible to experience, and its application is always hypothetical (comparable to the notion of force in classical mechanics, if one doesn't fix *a priori* the law of motion)"

It is demanded that the real state of a system S_2 be independent of a spatially separated system S_1. A well-known and often quoted statement of Einstein, amounting to what is termed 'Einstein locality' is:

"But on one supposition we should, in my opinion, absolutely hold fast: the real factual situation of the system S_2 is independent of what is done with the system S_1 which is spatially separated from the former"

It is time now to see what EPR attempted in their historic paper.

6. The EPR Paper

The aim was to show that quantum mechanics was incomplete. Since we wish to present their work in a ical manner (to hopefully orient and assist any reader of their paper), various key words will be underlined for emphasis. To achieve their objective, they introduced three notions or concepts, followed by two statements about them. The concepts are

(*a*) a complete theory,
(*b*) elements of physical reality (or epr),
(*c*) counterpart of epr in physical theory. (8)

None of these concepts is defined comprehensively with full meaning, because according to EPR, that much is not needed. Next come the two statements involving these concepts:

(1) A necessary condition for a theory to be complete is that every epr must have a counterpart in the theory.

Thus, as stated above, no sufficient conditions for completeness are given or attempted. What about epr's which play a role in statement (1)? Only a sufficient condition which can help us recognise some of them is given:

"The epr's cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements"

The sufficient condition constitutes statement (2):

(2) Sufficient condition to identify an epr: "If, without in any way disturbing a system, we can predict with certainty (...) the value of a physical quantity, then there exists an epr corresponding to this physical quantity"

So the scheme of ideas can be depicted thus:

(a) Complete theory	(b) epr's
No full definition, only a necessary Condition(1) involving all epr's	No general definition, only a sufficient Condition(2) to identify some of them

(c) Counterpart of epr in theory
No precise meaning, simply left to be inferred

It seems reasonable to infer that 'epr' and 'counterpart' together mean the assignment of a definite numerical value to the concerned physical quantity, under specified circumstances. The strategy now is the following: Come up with a situation in quantum mechanics, some system S in some state $|\psi\rangle$, such that using the sufficient condition (2) some epr's can be found; then show that according to quantum mechanics they cannot all have counterparts in the quantum description, i.e., condition (1) is not satisfied; thus conclude that quantum mechanics is incomplete.

The EPR paper dealt with position and momentum eigenstates for two particles moving in one dimension. Since these are not normalisable states, in an equally good version Bohm considered a total system $S = A+B$, where A and B are two spin$\frac{1}{2}$ particles moving in physical space. They are supposed to be initially close together, then move far apart in such a way that the total spin wave function is unchanged. The spin wave function $|\psi\rangle$ is chosen to be the singlet state which is invariant under all spatial rotations, and can therefore be written in (infinitely) many equivalent ways. We choose the following two expressions:

$$\begin{aligned} |\psi\rangle &= \frac{1}{\sqrt{2}}\{|z,+\rangle|z,-\rangle - |z,-\rangle|z,+\rangle\} \\ &= \frac{1}{\sqrt{2}}\{|x,+\rangle|x,-\rangle - |x,-\rangle|x,+\rangle\} \qquad (9) \end{aligned}$$

Here, in each term on the right hand side, the first factor is the state of A and the second factor that of B; for either particle, $|z,\pm\rangle$ are eigenstates of S_z with eigenvalues $\pm 1/2, |x,\pm\rangle$ are eigenstates of S_x with eigenvalues $\pm 1/2$. The situation can be depicted as follows:

$$\begin{array}{ccc} \times \longleftarrow & \bigcirc & \longrightarrow \times \\ A & |\psi\rangle \text{ formed here} & B \end{array} \qquad (10)$$

Now, in effect, EPR argue as follows: If we wish, we can measure $S_z^{(A)}$; the possible results are $\pm 1/2$; after the measurement the wave function collapses to $|z, \pm\rangle|z, \mp\rangle$; thus 'without disturbing B' we can infer that $S_z^{(B)}$ has value $\mp 1/2$; therefore $S_z^{(B)} = \mp 1/2$ is an epr. On the other hand, if we wish, we can start by measuring $S_x^{(A)}$, getting the result $\pm 1/2$; and then following a parallel line of reasoning, we conclude that $S_x^{(B)} = \mp 1/2$ is an epr. By locality, the 'real state' of B should be unaffected by what is measured at A. At A we can choose what to measure, either $S_z^{(A)}$ or $S_x^{(A)}$ but not both. Since we have shown the existence of two epr's for B, both $S_z^{(B)}$ and $S_x^{(B)}$ should have had definite values already. As they do not commute, quantum mechanics cannot account for this. Therefore quantum mechanics is incomplete.

One sees from the EPR argument that for 'an epr to have a counterpart in the theory' means 'for the concerned physical quantity to have a definite numerical value' ie., for the state to be the relevant eigenstate of the operator.

The expression for $|\psi\rangle$ in equation (9) *cannot* be written in the product form $|\varphi\rangle_A\,|\chi\rangle_B$, for any choices of $|\varphi\rangle$ and $|\chi\rangle$. Such states are said to be *entangled*.

Later accounts say that the EPR paper was drafted by Podolsky and not seen by Einstein in its final form. (For example, as a rule Einstein always spoke of the 'psi-function' never of the wavefunction!) He felt his views were not well presented. Here is his version of the incompleteness argument. As we saw earlier, one of his basic requirements was that the 'real state' of B should be independent of the spatially separated A. Then: quantum mechanics would be complete if and only if there

Quantum mechanics would be complete if and only if there is a one-to-one correspondence between real states and wavefunctions.

is a one-to-one correspondence between real states of B and wavefunctions ψ_B for B (upto overall phases, and limiting ourselves to pure states). For the composite system $S = A + B$ with spatially separated parts A and B, suppose the overall wavefunction ψ_{AB} is not a product but is entangled. Then the wavefunction we ascribe to B after a measurement on A followed by collapse depends on what is measured on A. Schematically:

$$\psi_{AB} \text{ for } S = A + B$$

$$\left\{ \begin{array}{l} \text{measure something on } A, \\ \text{use collapse, get } \psi_B \text{ for } B \\ \\ \text{assume incompatible} \\ \\ \text{measure something else on } A, \\ \text{use collapse, get } \psi'_B \text{ for } B \end{array} \right\} B \text{ is not disturbed} \quad (11)$$

The 'real state' of B is thus not represented by a unique wavefunction, independent of operations on A. The wavefunction for B, found via collapse, depends on what is measured on A which is far away. This nonuniqueness of the wavefunction assigned to B shows that quantum mechanics is incomplete. In effect, Einstein believed that, being spatially separated, A and B have their respective individual 'real states', which together determine the state of $S = A + B$.

For clarity let us repeat the two forms of the incompleteness argument:

EPR paper: we show the existence of two epr's for B but no ψ_B in quantum mechanics can accommodate both of them.

Einstein's version: B has one 'real state' but quantum mechanics does not give us a unique ψ_B to represent it.

(12)

Clearly, both forms involve making two mutually incompatible measurements on A. On this aspect, EPR admit:

"... one would not arrive at our conclusion if one insisted that two or more physical quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted"

Clearly they do not require this stronger condition while identifying two epr's for B – they imagine two mutually incompatible experiments being carried out on A, only one of which can actually be carried out at a given time. They however say: if one adopted the above stronger condition, only one epr can be identified for B, but it depends on what is measured at A which is far away, and this is unreasonable: "No reasonable definition of reality could be expected to permit this"

So the two options, both of which they find unacceptable, are these:

Weaker: Consider two mutually exclusive measurements on A; imagining either one or the other being carried out at some time, claim to have found two epr's for B; both should have counterparts in a complete theory; quantum mechanics has no such simultaneous eigenstates, so it is incomplete.

Stronger: Since the two measurements contemplated on A are incompatible, we can carry out only one or the other, then infer the existence of a corresponding epr for B. But B is far away from A, so an epr for it cannot depend on what is measured at A. It is unreasonable to permit such dependence.

7. Bohr's Response

As we mentioned in Section 1, Bohr's response was contained in a paper in the very next volume of the *Physical Review* in the same year, 1935, as EPR. According to Pais,

"(Bohr) did not believe that the EPR paper called for any change in the interpretation of quantum mechanics"
– *Pais*

"(Bohr) did not believe that the EPR paper called for any change in the interpretation of quantum mechanics"

Bohr's paper is not easy reading; here is an attempt to convey what he probably had in mind. Consistent with Pais' statement quoted above, we say: If $S = A + B$ is in an entangled (ie. nonproduct) state, there are no mutually independent (pure) states for A and B even if they are spatially far apart; separability of state with respect to subsystems does not hold in quantum mechanics. Large spatial separation of A and B may imply absence of physical interaction between them, but it does not imply existence of independent pure states for A and B. When ψ_{AB} is entangled, the pure state obtained for B via collapse after a measurement on A does depend on what is measured on A, and is to be used for making predictions about any (later) measurements made on B.

Thus quantum correlations between A and B in an entangled state ψ_{AB} are distance independent. This means that quantum mechanics is nonlocal at the wave function level. As parts of a total $S = A+B$, A and B do not always 'possess' independent individual wavefunctions. Nevertheless these correlations cannot be used to send messages. A knows the resulting state of B at the end of his measurement, but not B. In other words, even though the result of a measurement at A plus wavefunction collapse allows us to predict a certain measurement result at B with certainty, the former is uncertain and governed by probabilities; that uncertainty then remains for any prediction concerning B.

Quantum correlations are distance dependent.

Statistically speaking, quantum mechanics does obey locality, in the following precise sense. For any observable $\hat{B}$ of B, whether we make some measurement on A (and retain all results) or not, the expectation value is the same, namely:

$$\begin{aligned} \langle\hat{B}\rangle_{|\psi_{AB}\rangle} &= \mathrm{Tr}_B(\hat{B}\,\hat{\rho}_B), \\ \hat{\rho}_B &= \mathrm{Tr}_A\,|\psi_{AB}\rangle\langle\psi_{AB}|. \end{aligned} \qquad (13)$$

Here Tr_A is to be read as partial trace with respect to the subsystem A, which amounts to 'ignoring' A or treating all states of A on equal footing.

One is tempted to ask the question: where did EPR 'go wrong'? Why does quantum mechanics not abide by their 'innocent looking' conditions? The answer is that those conditions are not really so innocent! Bohm suggests that in addition to their two statements (1), (2) in Section 6, they made two more implicit assumptions:

(3) The world can be correctly analysed in terms of distinct and separately existing epr's;

(4) Every one of these epr's must be a counterpart of a precisely defined mathematical quantity appearing in a complete theory.

Here we see the word 'counterpart' again! Presumably the meaning of (4) is that if we have an epr, then some variable in the theory must have a corresponding numerical value. Of course, as Bohm says, quantum mechanics does not abide by (3) and (4); it does not allow us to work wholly with a set of dynamical variables always possessing definite numerical values.

One can even take the following attitude: given the singlet state $|\psi\rangle$ of equation(9) for the pair of spin1/2 particles $A+B$, no single unambiguous epr has been shown to exist for B. From this point of view, Einstein's view described in Section 6 seems preferable as a criticism of quantum mechanics.

One is finally left with the feeling that EPR's criticism reduces to their not liking quantum mechanics, as it does not agree with their prejudices about any theory.

The inescapable conclusion is that quantum mechanics violates local realism.

8. Bell's Analysis

We saw in Section 5 that Einstein insisted upon both locality and objective reality as general requirements for any physical theory. Their combination is called 'local realism' and it was analysed in precise fashion by John Bell. He found an inequality which any local realist theory should obey, but which quantum mechanics does not. The inescapable conclusion is that quantum mechanics violates local realism.

Here is Bell's argument, referring again to the two spin 1/2 system $A + B$, assumed to be in the singlet state $|\psi\rangle$ of equation (9). Quantum mechanics does not allow us to imagine that any spin component of A (or of B) has a definite value, in the absence of some measurement. Imagine now that quantum mechanics can be extended to, or embedded within, some more encompassing theory involving some hidden variables λ. If λ were known, we suppose that we could then say: for any three-dimensional unit vectors $\underline{a}$ and $\underline{b}$:

$$\begin{aligned} \underline{a} \;\; \underline{\sigma}^{(A)} \; \underline{\text{has}} \text{ the numerical value } A(\underline{a}, \lambda) &= \pm 1, \\ \underline{b} \;\; \underline{\sigma}^{(B)} \; \underline{\text{has}} \text{ the numerical value } B(\underline{b}, \lambda) &= \pm 1. \end{aligned} \quad (14)$$

Realism is expressed by the possibility of assigning definite numerical values $A(\underline{a}, \lambda), B(\underline{b}, \lambda)$ to $\underline{a} \;\; \underline{\sigma}^{(A)}, \underline{b} \;\; \underline{\sigma}^{(B)}$ (for all choices of $\underline{a}, \underline{b}$) if λ were known. Locality is expressed by the $\underline{b}$-independence of $A(\underline{a}, \lambda)$ and the $\underline{a}$-independence of $B(\underline{b}, \lambda)$.

Let $\rho(\lambda)$ be the probability distribution of λ, possibly dependent on $|\psi\rangle$.

Then the correlation between components of A-spin and B-spin in general directions is

$$P(\underline{a}, \underline{b}) = \int d\lambda \; \rho(\lambda) \; A(\underline{a}, \lambda) \; B(\underline{b}, \lambda). \quad (15)$$

Take four directions $\underline{a}, \underline{a}', \; \underline{b}, \; \underline{b}'$ to get two preliminary inequalities:

$$\begin{aligned}
&|P(\underline{a},\underline{b}) - P(\underline{a},\underline{b}')| = \\
&|\int d\lambda\ \rho(\lambda)\ A(\underline{a},\lambda)(B(\underline{b},\lambda) - B(\underline{b}',\lambda))| \\
&\leq \int d\lambda\ \rho(\lambda)|B(\underline{b},\lambda) - B(\underline{b}',\lambda)|; \\
&|P(\underline{a}',\underline{b}) + P(\underline{a}',\underline{b}')| = \\
&|\int d\lambda\ \rho(\lambda)\ A(\underline{a}',\lambda)(B(\underline{b},\lambda) + B(\underline{b}'\ \lambda))| \\
&\leq \int d\lambda\ \rho(\lambda)\ |B(\underline{b},\lambda) + B(\underline{b}',\lambda)|. \qquad (16)
\end{aligned}$$

Many experiments over the years, with increasing precision, have shown violation of local realist inequalities, and in agreement with quantum mechanics. Nature does not respect local realism!

Adding these gives the Bell inequality:

$$|P(\underline{a},\underline{b}) - P(\underline{a},\underline{b}')| + |P(\underline{a}',\underline{b}) + P(\underline{a}',\underline{b}'| \leq \int d\lambda\ \rho(\lambda)\times$$

$$\{|B(\underline{b},\lambda) - B(\underline{b}'\ \lambda)| + |B(\underline{b},\lambda) + B(\underline{b}'\ \lambda)|\} = 2, \quad (17)$$

the final result following from the fact that if one term within the curly brackets is 0 the other is 2 and conversely. Thus local realism implemented via hidden variables entails

$$|P(\underline{a},\underline{b}) - P(\underline{a},\underline{b}')| + |P(\underline{a}',b) + P(\underline{a}',\underline{b}')| \leq 2. \quad (18)$$

Now in the singlet state (9) quantum mechanics gives the value

$$P_{QM}(\underline{a},\underline{b}) = -\underline{a}\ \ \underline{b} \quad (19)$$

We choose $\underline{a},\underline{b},\underline{a}',\underline{b}'$ to be coplanar with $\underline{b}$ making an angle $\pi/4$ with $\underline{a},\underline{a}'$ a further $\pi/4$ with $\underline{b}$ (and so $\pi/2$ with $\underline{a}$), and $\underline{b}'$ another $\pi/4$ with $\underline{a}'$ (and so $3\pi/4, \pi/2$ with $\underline{a},\underline{b}$ respectively). Then the left hand side of (18) is $2\sqrt{2}$, in violation of that inequality.

Many experiments over the years, with increasing precision, have shown violation of such local realist inequalities, and in agreement with quantum mechanics. Nature does not respect local realism!

9. The Hardy State

Even realism is untenable in quantum mechanics! We know this already from the discussion of Section 7, but here is another striking illustration, due to Lucien Hardy. We consider again a pair $A+B$ of spin1/2 particles. For each, we contemplate measurements of one of two non-commuting variables, σ_x and σ_z. We search for a pure state $|\psi\rangle$ obeying three conditions:

$$\begin{array}{ll} (i) & \text{If measurement of } \sigma_z^{(A)} \text{ yields } +1, \\ & \text{it must then yield } \sigma_z^{(B)} = +1; \\ (ii) & \text{If measurement of } \sigma_z^{(B)} \text{ yields } +1, \\ & \text{it must then yield } \sigma_x^{(A)} = +1; \\ (iii) & \text{If measurement of } \sigma_x^{(A)} \text{ yields } +1, \\ & \text{it must then yield } \sigma_x^{(B)} = +1; \end{array} \tag{20}$$

In an obvious notation, these three conditions lead to the following three expressions for $|\psi\rangle$:

$$\begin{aligned} |\psi\rangle &= N_1(|z+\rangle|z+\rangle + |z-\rangle|\alpha\rangle) \\ &= N_2(|x+\rangle|z+\rangle + |\beta\rangle|z-\rangle) \\ &= N_3(|x+\rangle|x+\rangle + |x-\rangle|\gamma\rangle). \end{aligned} \tag{21}$$

In each term, the first/second factor is an A state/B state; $|z\pm\rangle$ are σ_z eigenstates; $|x\pm\rangle$ are σ_x eigenstates,

$$|x\pm\rangle = \frac{1}{\sqrt{2}}(|z+\rangle \pm |z-\rangle); \tag{22}$$

N's are normalization factors; $|\alpha\rangle, |\gamma\rangle$ are some B-states, and $|\beta\rangle$ is some A-state. Simple algebra then gives:

$$\begin{aligned} |\alpha\rangle &= |z+\rangle + 2|z-\rangle, \\ |\beta\rangle &= \sqrt{2}|z-\rangle, \\ |\gamma\rangle &= -\frac{1}{\sqrt{2}}\,|z-\rangle, \\ N_1 = \frac{1}{\sqrt{6}}, N_2 &= \frac{1}{\sqrt{3}},\ N_3 = \sqrt{\frac{2}{3}}. \end{aligned} \tag{23}$$

We have as a consequence:

$$\{\langle z+|\langle x-|\}|\psi\rangle = \frac{1}{2\sqrt{3}}. \tag{24}$$

Classical realism would mean, on the basis of (20): if $\sigma_z^{(A)}$ definitely has the value +1, then $\sigma_z^{(B)}$ definitely has value +1, then $\sigma_x^{(A)}$ has definitely the value +1, then $\sigma_x^{(B)}$ also definitely has value +1. But from quantum mechanics we see that equation (24) says that there is an 8.33% probability that $\sigma_z^{(A)} = +1$ and $\sigma_x^{(B)} = -1$. This is an illustration of the loss or lack of realism in quantum mechanics.

An even more striking illustration not involving any probabilities at all is due to Greenberger, Horne and Zeilinger, and involves a set of three spin half particles. But let us leave that to the curious reader!

10. Concluding Comments

One should remember that the EPR paper was written seventy years ago, when quantum mechanics was barely a decade old. Since that time, by and large physicists have grown accustomed to the counter-intuitive features of quantum phenomena. Even though our account of EPR has been intentionally critical, it must be admitted that it has highlighted specific features of quantum mechanics – the issues of realism, locality and entanglement. It immediately inspired Schrödinger's ideas on entanglement, which is a reflection of the enormous richness contained in the tensor product rule $\mathcal{H}_{A+B} = \mathcal{H}_A \otimes \mathcal{H}_B$ for composite system state spaces. Today entanglement is the key resource for quantum computation, and for other quantum information processes. The ideas of realism and locality which Einstein would never give up led to Bohm's efforts to find an almost classical interpretation of quantum mechanics, and then to Bell's incisive analysis of the full implications of local realism. Now we know that quantum correlations are very subtle, and go beyond classical limits even without involving large spatial separations. Thus quantum correlations at the wave function level are distance-independent. Thus all this has resulted from the brief EPR paper.

We hope that young readers of our account will feel confident in looking at the literature in this area of physics with a good sense of direction to guide them.

Acknowledgements: We thank Vidyanand Nanjundiah for a critical reading of the manuscript and many useful suggestions.

Address for Correspondence
S Chaturvedi
School of Physics
University of Hyderabad
Hyderabad 500 046, India.
Email:scsp@uohyd.ernet.in

N Mukunda
Centre for High Energy Physics
Indian Institute of Science
Bangalore 560 012, India.
Email:
nmukunda@cts.iisc.ernet.in

R Simon
Institute of Mathematical Sciences
Chennai 600 113, India.
Email:simon@imsc.res.in

Suggested Reading

[1] J S Bell, *Speakable and unspeakable in quantum mechanics*, Cambridge University Press, Cambridge, 1987.

[2] D Bohm, *Quantum Theory*, Dover Publications, 1989; *The Undivided Universe*, Routledge, UK, 1995.

[3] Dipankar Home, *Conceptual Foundations of Quantum Physics*, Plenum Press, New York and London, 1997.

[4] A Pais, *Niels Bohr's Times: In Physics, Philosophy, and Polity*, Oxford, 1991.

[5] L Hardy, *Phys. Rev. Lett.*, Vol. 71, p.1665, 1993.

[6] Arvind, The epr paradox: Einstein scrutinises quantum mechanics, *Resonance*, Vol.5, No.4, p.28, 2000.

[7] R Simon, From Shannon to Quantum Information Science : Ideas and Techniques, *Resonance*, Vol.7, No.2, p.66, 2002.

"Einstein was unable to accept as final the wholly unorthodox mathematical formulation of Planck's quantum theory,...,since it did not correspond to his philosophical conceptions of the task of the exact sciences. He felt it disturbing that natural laws should have to relate not to objective processes but to the possibility or probability of such processes"

–*Werner Heisenberg*
in '*Planck's Discovery and Atomic Theory*'