Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.

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Solution to
On a Use of Normal Distribution

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Problem

For positive integers n, consider the quantities A_n and B_n defined by

$$A_n = \int_{-1}^{1} \cos^{2n} \frac{\pi x}{2}\, dx,$$

$$B_n = \frac{1}{A_n} \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^{2n} \frac{\pi x}{2}\, dx. \qquad (1)$$

It is easy to anticipate that $A_n \to 0$ when $n \to \infty$; some sample values are displayed below, computed using *Mathematica*.

$$A_{100} \approx 0.1127, A_{200} \approx 0.0797,$$

$$A_{400} \approx 0.0564, A_{1000} \approx 0.0357. \qquad (2)$$

Keywords
Normal distribution.

Since the range of integration in the integral defining B_n steadily diminishes, we might also guess that $B_n \to 0$

as $n \to \infty$. But the figures tell us otherwise; indeed we find that

$$B_{100} \approx 0.9742, B_{200} \approx 0.9739,$$

$$B_{400} \approx 0.9738, B_{1000} \approx 0.9737. \qquad (3)$$

The figures suggest that B_n converges to a non-zero value (approximately 0.974) as $n \to \infty$. How may this be explained? What is the limiting value, and can it be expressed in closed form?

Solution

The following may be verified:

$$A_1 = 1, \quad A_2 = \frac{3}{4}, \quad A_3 = \frac{5}{8}, \quad A_4 = \frac{35}{64}, \quad A_5 = \frac{63}{128}. \qquad (4)$$

It is easy to obtain an exact formula for A_n. Integrating by parts, we get

$$\frac{A_n}{A_{n-1}} = \frac{2n-1}{2n} \quad (\text{for } n > 1), \qquad (5)$$

and since $A_1 = 1$, this leads to:

$$A_n = \frac{1}{2^{2n-1}} 2n()n. \qquad (6)$$

Using Stirling's approximation, $n! \approx \sqrt{2\pi n}\,(n/e)^n$, we get the useful approximation

$$A_n \approx \frac{2}{\sqrt{n\pi}}. \qquad (7)$$

which tells us that A_n is roughly proportional to $1/\sqrt{n}$ for large n. This is what our data also reveal; e.g., we see that A_{400} is roughly half of A_{100}.

For $n \gg 1$, the range of integration in the integral defining B_n is a narrow interval surrounding 0, so we may use the approximation $\cos x \approx 1 - x^2/2$. We get, for $n \gg 1$:

$$\begin{aligned} \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^{2n} \frac{\pi x}{2}\, dx &\approx \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \left(1 - \frac{\pi^2 x^2}{8}\right)^{2n} dx \\ &= \frac{1}{\sqrt{n}} \int_{-1}^{1} \left(1 - \frac{\pi^2 u^2}{8n}\right)^{2n} du \\ &\approx \frac{1}{\sqrt{n}} \int_{-1}^{1} e^{-\pi^2 u^2/4}\, du. \qquad (8) \end{aligned}$$

Making another change of variable, $v = \pi u/2$, we get

$$\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^{2n} \frac{\pi x}{2}\, dx \approx \frac{2}{\pi\sqrt{n}} \int_{-\pi/2}^{\pi/2} e^{-v^2}\, dv. \qquad (9)$$

Since $A_n \approx 2/\sqrt{\pi n}$ for $n \gg 1$, we get, for $n \gg 1$,

$$B_n \approx \sqrt{\pi} \int_{-\pi/2}^{\pi/2} e^{-v^2}\, dv = \frac{\int_{-\pi/2}^{\pi/2} e^{-v^2}\, dv}{\int_{\infty}^{\infty} e^{-v^2}\, dv}. \qquad (10)$$

We have found our answer: the limiting value of B_n is

$$\frac{\int_{-\pi/2}^{\pi/2} e^{-v^2}\, dv}{\int_{\infty}^{\infty} e^{-v^2}\, dv} \approx 0.97368, \qquad (11)$$

agreeing exactly with our computations.

Remark. Inasmuch as the answer has been found in terms of the density function of the normal distribution, it would seem that we should be able to get the answer through a probability-based argument. Does the reader see any such solution?