Think It Over

This section of *Resonance* presents thought-provoking questions, and discusses answers a few months later. Readers are invited to send new questions, solutions to old ones and comments, to 'Think It Over', *Resonance*, Indian Academy of Sciences, Bangalore 560 080. Items illustrating ideas and concepts will generally be chosen.


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Solution to
On a Use of Normal Distribution

Problem

For positive integers \( n \), consider the quantities \( A_n \) and \( B_n \) defined by

\[
A_n = \int_{-1}^{1} \cos^2 n \frac{\pi x}{2} \, dx,
\]

\[
B_n = \frac{1}{A_n} \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^2 n \frac{\pi x}{2} \, dx.
\]  

(1)

It is easy to anticipate that \( A_n \to 0 \) when \( n \to \infty \); some sample values are displayed below, computed using *Mathematica*.

\[
A_{100} \approx 0.1127, \quad A_{200} \approx 0.0797,
\]

\[
A_{400} \approx 0.0564, \quad A_{1000} \approx 0.0357.
\]  

(2)

Since the range of integration in the integral defining \( B_n \) steadily diminishes, we might also guess that \( B_n \to 0 \).
as \( n \to \infty \). But the figures tell us otherwise; indeed we find that

\[
B_{100} \approx 0.9742, \quad B_{200} \approx 0.9739,
\]

\[
B_{400} \approx 0.9738, \quad B_{1000} \approx 0.9737. \tag{3}
\]

The figures suggest that \( B_n \) converges to a non-zero value (approximately 0.974) as \( n \to \infty \). How may this be explained? What is the limiting value, and can it be expressed in closed form?

**Solution**

The following may be verified:

\[
A_1 = 1, \quad A_2 = \frac{3}{4}, \quad A_3 = \frac{5}{8}, \quad A_4 = \frac{35}{64}, \quad A_5 = \frac{63}{128}. \tag{4}
\]

It is easy to obtain an exact formula for \( A_n \). Integrating by parts, we get

\[
\frac{A_n}{A_{n-1}} = \frac{2n - 1}{2n} \quad \text{(for} \ n > 1, \text{)} \tag{5}
\]

and since \( A_1 = 1 \), this leads to:

\[
A_n = \frac{1}{2^{2n-1}} 2n(n). \tag{6}
\]

Using Stirling's approximation, \( n! \approx \sqrt{2\pi n} (n/e)^n \), we get the useful approximation

\[
A_n \approx \frac{2}{\sqrt{n\pi}}. \tag{7}
\]

which tells us that \( A_n \) is roughly proportional to \( 1/\sqrt{n} \) for large \( n \). This is what our data also reveal; e.g., we see that \( A_{400} \) is roughly half of \( A_{100} \).
For \( n \gg 1 \), the range of integration in the integral defining \( B_n \) is a narrow interval surrounding 0, so we may use the approximation \( \cos x \approx 1 - \frac{x^2}{2} \). We get, for \( n \gg 1 \):

\[
\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^{2n} \frac{\pi x}{2} \, dx \approx \int_{-1/\sqrt{n}}^{1/\sqrt{n}} \left( 1 - \frac{\pi^2 x^2}{8} \right)^{2n} \, dx
\]

\[
= \frac{1}{\sqrt{n}} \int_{-1}^{1} \left( 1 - \frac{\pi^2 u^2}{8n} \right)^{2n} \, du
\]

\[
\approx \frac{1}{\sqrt{n}} \int_{-1}^{1} e^{-\frac{\pi^2 u^2}{4}} \, du. \tag{8}
\]

Making another change of variable, \( v = \frac{\pi u}{2} \), we get

\[
\int_{-1/\sqrt{n}}^{1/\sqrt{n}} \cos^{2n} \frac{\pi x}{2} \, dx \approx \frac{2}{\pi \sqrt{n}} \int_{-\pi/2}^{\pi/2} e^{-v^2} \, dv. \tag{9}
\]

Since \( A_n \approx \frac{2}{\sqrt{\pi n}} \) for \( n \gg 1 \), we get, for \( n \gg 1 \),

\[
B_n \approx \sqrt{\pi} \int_{-\pi/2}^{\pi/2} e^{-v^2} \, dv = \frac{\int_{-\pi/2}^{\pi/2} e^{-v^2} \, dv}{\int_{\infty}^{\infty} e^{-v^2} \, dv}. \tag{10}
\]

We have found our answer: the limiting value of \( B_n \) is

\[
\frac{\int_{-\pi/2}^{\pi/2} e^{-v^2} \, dv}{\int_{\infty}^{\infty} e^{-v^2} \, dv} \approx 0.97368, \tag{11}
\]

agreeing exactly with our computations.

**Remark.** Inasmuch as the answer has been found in terms of the density function of the normal distribution, it would seem that we should be able to get the answer through a probability-based argument. Does the reader see any such solution?