Foundation of Basic Arithmetic

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Part 1. Foundation of Basic Arithmetic, *Resonance*, Vol.11, No.1, pp.8-20, 2006.

Keywords
Different bases for arithmetic, place value system.

This is the second of a series of articles, in which we demonstrate the utility of the place value system of representing numbers.

If we use base β, we need β digits for zero, one, two, , $\beta - 1$. For example, in base ten, our digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For $\beta > 10$, we would have to invent more digits. A number n may be represented in any given base. The number nine, which is represented in base ten as 9, has the representation 1001 in base two, i.e., $9 = (1001)_2$. This means that $9 = 1 \;\; 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1$. The representation of nine in base three is 100 for $9 = 1 \;\; 3^2 + 0 \;\; 3 + 0$.

Once we have chosen a particular base it is as easy to do simple computations such as adding and multiplying as in our everyday number system with base ten. Before doing arithmetic in a different base, let us first examine the process for doing arithmetic in base ten.

Example 1. Adding 369 to 5184 involves the following steps:

1. We first align the numbers so that the digits representing units are in the same column on the right:

$$\begin{array}{r} 3\,6\,9 \\ +\,5\,1\,8\,4 \\ \hline \end{array}$$

2. We consider first the units (i.e. the digits on the far right). Since the sum of the units is greater than ten (base in which we are working), we place the carry over one above the next column, the column representing groups of ten, and write down

the digit 3 for leftover units below the summation line.

$$
\begin{array}{r}
{\scriptstyle 1} \\
3\,6\,9 \\
+\,5\,1\,8\,4 \\
\hline
3
\end{array}
$$

3. We do the same now for the column representing tens, and take the carry over to the next column representing hundreds if the sum of the digits is greater than ten. In essence, because we are adding in the tens column, we are actually adding 60, 80 and the one ten which we carried over from the units.

$$
\begin{array}{r}
{\scriptstyle 1\ 1} \\
3\,6\,9 \\
+\,5\,1\,8\,4 \\
\hline
5\,3
\end{array}
$$

4. We continue this process until we reach the last column to get

$$
\begin{array}{r}
{\scriptstyle 0\ 1\ 1} \\
3\,6\,9 \\
+\,5\,1\,8\,4 \\
\hline
5\,5\,5\,3
\end{array}
$$

Hence $369 + 5184 = 5553$.

It is possible to do arithmetic in any base as easily as in base ten, the base we are so accustomed to.

Example 2. We want to add the numbers 2324 and 103 with base $\beta = five$. First note that our representation is not in base ten so 103 is not one hundred and three. In fact, 103 is $1 \quad 5^2 + 0 \quad 5 + 3$. Hence this number is to be read as 'one–zero–three'. Also note that we need only the digits 0, 1, 2, 3, 4 and discard the remaining 5, 6, 7 8, 9. Again, to add

1. We align these two numbers as shown below with the units digits in the right most column.

$$\begin{array}{r} 2\,3\,2\,4 \\ +\quad 1\,0\,3 \\ \hline \end{array}$$

2. Starting with the units column, we add the digits. Since the sum is seven, two greater than our base five, we carry one to the next column, and write down the digit for two (surplus from one group of five) as shown.

$$\begin{array}{r} {\scriptstyle 1} \\ 2\,3\,2\,4 \\ +\quad 1\,0\,3 \\ \hline 2 \end{array}$$

3. Continuing this process until we reach the last column, we get

$$\begin{array}{r} {\scriptstyle 1} \\ 2\,3\,2\,4 \\ +\quad 1\,0\,3 \\ \hline 2\,4\,3\,2 \end{array}$$

Multiplication and division can also be done in any base in a similar manner. A multiplication table for base β is $\beta \times \beta$ if the multiplication by 0 is included in the table. If multiplication by zero is ignored, as it always results in zero, the table is $(\beta - 1) \times (\beta - 1)$. One memorizes the traditional multiplication table for base ten in elementary school.

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

How do we read this table? To find the product 6×7, we look at the entry in the 6th row and 7th column. Because multiplication is commutative, that is $a \times b = b \times a$ (why?) this product is the same as the entry in the 7th row and 6th column. Thus we actually need only the part of the table shown in bold face.

Tables for other bases are constructed in the same way. For base five, the multiplication table would appear as below.

	1	**2**	**3**	**4**
1	**1**	**2**	**3**	**4**
2		**4**	**11**	**13**
3			**14**	**22**
4				**31**

If the base is two the table is trivial.

	0	**1**
0	**0**	**0**
1		**1**

If $\beta \leq$ ten, we may use the first β digits 0, 1, 2, $\beta - 1$ from 0, 1, 2, , 9 and discard the remaining digits. But if β is more than ten, the ten digits 0, 1, 2, , 9 are not enough. We must invent symbols for the extra digits. Let us consider base twelve. We will need two more digits. We can choose them arbitrarily. Let τ represent ten and let ε represent eleven. Now we can construct the multiplication table for base twelve.

We can now use this table to multiply any two numbers in base twelve. The procedure is similar to that of adding.

	1	2	3	4	5	6	7	8	9	τ	ε
1	1	2	3	4	5	6	7	8	9	τ	ε
2		4	6	8	τ	10	12	14	16	18	1τ
3			9	10	13	16	19	20	23	26	29
4				14	18	20	24	28	30	34	38
5					21	26	2ε	34	39	42	47
6						30	36	40	46	50	56
7							41	48	53	5τ	65
8								54	60	68	74
9									69	76	83
τ										84	92
ε											$\tau 1$

Example 3. To illustrate, we will multiply $4\tau 3$ by 1ε.

1. First align these two numbers with the units digits in the far right column.

$$\begin{array}{r} 4\ \tau\ 3 \\ \times\quad 1\ \varepsilon \\ \hline \end{array}$$

2. Starting with the units column, we multiply the digits. Using the table above, we can multiply any two digits. We start at the units place and find that $3 \times \varepsilon = 29$ (it is not twenty-nine, but 'two–nine' in base twelve, i.e., two dozens plus nine). We place the units digit 9 under the product bar and carry the digit 2 to the next column as shown below.

$$\begin{array}{r} {\scriptstyle 2} \\ 4\ \tau\ 3 \\ \times\quad 1\ \varepsilon \\ \hline 9 \end{array}$$

3. Just as in 'usual' multiplication, the digits that are carried over to the next column must be added into the product of digits in the next step. In our

problem, we must now multiply τ by ε and add the 2 we carried over from the units column. From the table, $\tau \times \varepsilon = 92$. Add 2 to get 94. Once again, place the 4 under the product bar and carry over 9. By multiplying 4 by ε, we get 38, but now 9 must be added to the product. We use the procedure outlined above for addition to get $38 + 9 = 45$. Because we have no more digits left to multiply, we place 45 under the product bar as shown below.

$$\begin{array}{r} {\scriptstyle 9\,2\;} \\ 4\,\tau\,3 \\ \times\;\; 1\,\varepsilon \\ \hline 4\,5\,4\,9 \end{array}$$

4. To begin multiplying with the second digit, we place a zero place holder for obvious reasons under the units column and proceed with the multiplication as shown below.

$$\begin{array}{r} {\scriptstyle 9\,2\;} \\ 4\,\tau\,3 \\ \times\;\; 1\,\varepsilon \\ \hline 4\,5\,4\,9 \\ 4\,\tau\,3\,0 \end{array}$$

5. Now the two rows below the product bar are added together as explained earlier to get the final answer.

$$\begin{array}{r} {\scriptstyle 1\;\;\;\;\;\;} \\ 4\,5\,4\,9 \\ +\,4\,\tau\,3\,0 \\ \hline 9\,3\,7\,9 \end{array}$$

Long division is done in a similar manner.

Example 4. Let β = seven. We want to find the quotient and the remainder when we divide the number whose representation in base seven is 51643 by ten, i.e. by 13.

$$
\begin{array}{r@{\,}l}
 & 3460 \\
13 & \overline{)\,51643} \\
 & -42 \\
 & \,66 \\
 & -55 \\
 & 114 \\
 & -114 \\
 & 03 \\
 & -0 \\
 & 3
\end{array}
$$

So the quotient $q = 3460$, and the remainder $r = 3$.

Base Change

Our decimal number system is a human invention, it is not 'God given' (In some sense it is. We did not choose how many fingers a person could have.) The only hurdle is that we are so well trained in decimal numerals that it is difficult to think otherwise. As a matter of fact, when we see 52, in say base seven, we are tempted to read it as fifty-two (five tens and two) while it is actually $5 \cdot 7 + 2 =$ thirty-seven. The obstacles are not conceptual, but habitual and linguistic ones. The names of the numbers themselves are decimal based. The number whose decimal representation is 24, i.e. two tens and four, is called twenty-four.

During the transmission of our decimal number system from India to the West, different bases and our terminology got a little bit mixed up. The twenty four spoke wheel on the official seal of Emperor Ashoka (273-237 BC) suggests that base twelve may also have been used in India at that time. It is more certain that we inherited sixty minutes in an hour from the Babylonians.

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In English, two peculiar words, *eleven* and *twelve*, seem to be left over from a dozen after completing a group of ten. But after twelve we revert back to thirteen (three and ten), fourteen and so on. Strictly speaking, in our decimal system eleven and twelve could have been more appropriately called something like oneteen, twoteen. It may be noted that while in English these two words are anomalies, in Indian languages as well as in Latin, the words for eleven and twelve follow the same pattern as thirteen, fourteen and so on. In decimal based English, twenty means two tens. The Welsh used base nine, hence the word for eighteen (eight and ten) in Welsh is deu naw (two nines). Had we used base six, the word for eight would probably be twix (two and six). For more see [1].

Before working out some examples for *base change*, i.e. to change representation of a number given in one base to that in another one, the reader is reminded to read a representation, say 63 in base seven, simply as six three and not sixty-three, for in base seven 63 is six sevens plus three and not six tens plus three. If a base is not specified, it is understood to be our own decimal one.

Example 5.

Let $n = 101$, that is, one hundred and one. Suppose we want to express n in binary representation. To find digits, starting with the unit digit, all we need to do is to perform long division (in decimal system) with $\beta = 2$ repeatedly. The remainders are the digits:

$$\begin{aligned} 101 &= 50 \quad 2+1 \\ 50 &= 25 \quad 2+0 \\ 25 &= 12 \quad 2+1 \\ 12 &= 6 \quad 2+0 \\ 6 &= 3 \quad 2+0 \\ 3 &= 1 \quad 2+1 \\ 1 &= 0 \quad 2+1. \end{aligned}$$

Hence $$\begin{aligned} n &= (d_r \quad . d_2 d_1 d_0)_2 \\ &= (1100101)_2 \end{aligned}$$

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Example 6.

Let us find the decimal representation of the number n whose representation in base seven is 51643.

To do this we need to apply the division algorithm repeatedly to find the digits. For example, d_0 is given as the remainder of n on division by ten, that is by $\tau = 13$ (in base seven):

$$n = q_1\tau + d_0 \quad (d_0 < \tau).$$

The next digit d_1 is the remainder of q_1 on dividing by τ:

$$q_1 = q_2\tau + d_1$$

and so on.

But we now have to do the arithmetic in base seven. Using the calculation from Example 4 we have

$$51643 = q_1 \quad 13 + d_0$$

with $q_1 = 3460$ and $d_0 = 3$. Similarly,

$$\begin{aligned} 3460 &= 240 \quad 13 + 10 \quad &&\text{(Note that 10 in base seven} \\ 240 &= 15 \quad 13 + 6 &&\text{is seven.)} \\ 15 &= 1 \quad 13 + 2 \\ 1 &= 0 \quad 13 + 1 \end{aligned}$$

Hence $n = 12673$, in our decimal representation.

Fractions

When we write 532.64 we mean it is in base τ (for ten) the number

$$5\tau^2 + 3\tau + 2 + \frac{6}{\tau} + \frac{4}{\tau^2}.$$

The part

$$.64 = \frac{6}{\tau} + \frac{4}{\tau^2}$$

Exercises

1. Suppose $\beta =$ seven. Multiply 615.243 with 531.264.

2. Suppose in any base $\beta > 1$, the number $\alpha = 0.d_1 \cdots d_r \cdots$, where the block $d_1 \cdots d_r$ of r digits repeats forever. Show that α is rational, that is, a ratio of two whole numbers.

   Hint. Look at the number β^r

represents the *fractional part* of this number. If we are using $\beta =$ five, 402.31 would represent the number

$$4 \quad 5^2 + 0 \quad 5 + 2 + \frac{3}{5} + \frac{1}{5^2}.$$

Arithmetic involving fractions is carried out in the same manner. For example, in base five, to multiply .4 by .3, we would proceed as follows:

$$\begin{aligned} .4 \times .3 &= \frac{4}{5} \times \frac{3}{5} = \frac{4 \times 3}{5^2} \\ &= \frac{2 \quad 5 + 2}{5^2} = \frac{2}{5} + \frac{2}{5^2} \\ &= .22 \end{aligned}$$

In shorthand notation we have:

$$\begin{array}{r} 0.4 \\ \times \;\; 0.3 \\ \hline 0.22 \end{array}$$

Remarks

Binary representation (base two) is used as the 'machine language' of computers. The reason is that the digits 0 and 1 can be used as on and off signals, instructions for the circuitry in the computer. The disadvantage of using a smaller base such as two is that we need a longer string of digits to represent a number, e.g. even a single digit number such as 9 (in our decimal system) has the binary representation 1001. However, if the base is too big, say sixty, we would need sixty symbols, one for each digit, to represent numbers.

It is just a coincidence that the number of fingers on our two hands is just about the right size for a base. It is not too big or too small. On the other hand, arguably twelve would make a better base because it has many factors. Our notion of dozen is most likely related to this fact. A dozen can be divided into two, three, four, or six equal parts whereas ten has only two non-trivial factors. For example, canned drinks are packed in dozens as 3×4 rather than a skinny package of 2×5. Also, traces of the Babylonian use of base sixty remain imbedded in our culture with the use of 360 degrees in the circle and 60 minutes. It is convenient to use base sixty for measuring time and angles because we may want to have half, third or quarter of an hour. If we had ten minutes in an hour, imagine how many minutes would make a third of an hour! To express a number of minutes as a percentage of an hour, the calculation involves base change.

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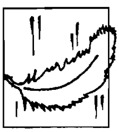
Suggested Reading

[1] K Menninger, *Number Words and Number Symbols*, Dover 1992.



36	144	72	108
216	864	252	288
576	2304	432	648
1286	3024	1152	1458
3456	5184	2592	3168

Some Gematrian Numbers.

From *Internet*