Foundation of Basic Arithmetic

Jasbir S Chahal

Jasbir S Chahal is a professor of mathematics at Brigham Young University, Provo, Utah, USA. His area of research is number theory.

This is the first of a series of articles, in which we shall briefly outline the early evolution of numerals as we currently know it.

During roughly the period 3000–1500 BC, the world's first four great civilizations flourished along river valleys. These are

1. The Nile Valley of Egypt,
2. Mesopotamia, meaning the land between the two rivers (Tigris and Euphrates) also called Babylonia, and now Iraq,
3. The Indus Valley (Punjab and Sind) of the Indian subcontinent, and
4. The Huang He Valley of China.

In these agrarian civilizations a minimal knowledge of mathematics was essential in building waterways, measuring the land and weighing the harvest for tax collection. To watch for the seasons, it was necessary to establish a calendar based on observing heavenly bodies. An understanding of geometry was needed to build palaces, pyramids, and tombs. Thus it is fairly safe to say that these civilizations had a functional knowledge of the rudimentary concepts of arithmetic, astronomy and geometry. In this article, we shall discuss the evolution of techniques for representing numbers and doing basic arithmetic.

Keywords
Roman, Egyptian and Chinese numerals, numerals of native Americans, Hindu-Arabic numerals, Quipu, Yupana, place-value system.

These societies had to devise ways to represent (whole) numbers, which would be amenable to calculations. Although the basic idea in every case was that of grouping, some were far superior to others, as can be seen by comparing the Roman numerals to the current decimal number system. As a prelude to doing arithmetic in an arbitrary base, it is illuminating to look at this basic notion of grouping.

The concept of numbers, or what we now call natural numbers, must be as old as any human civilization. For example, we would expect that a woman would be able to comprehend how many children she has given birth to. Experiments with birds show that parrots and ravens can compare the number of dots up to six.

Experiments with birds show that parrots and ravens can compare the number of dots up to six.

One way to define a number is that it is a count. The most natural way to record a count is by bars with one bar for each count. In fact, this way of recording a count is still common in many parts of the world, especially with carpenters and farmers.

When the count is large, it is quite inconvenient to keep track of these bars. To overcome this difficulty, it is common to cross every four marks with a fifth one. For example, the number seventeen can be represented by

The idea is to group the bars into groups of five. Nothing is special about the size of the group. In our example the size of the group is five, but we could have as easily chosen six for the group size. However, once we pick the group size we must remain consistent. As the count increases and even the number of groups becomes unmanageable, we can continue with this idea of 'grouping' by organizing groups of groups. So at a glance we can see that the following represents sixty-three:

In this example we have two groups of five fives, two groups of five, and three singles. A shorthand way of writing this tally can be accomplished by representing groups by symbols. Let a single bar | denote a unit, $\triangle$ denote a group of five, and ∇ denote a group of five fives; i.e., twenty-five. Then our tally would appear (with larger, or more important groups appearing first) as

$$\nabla\nabla\triangle\triangle|||$$

The fundamental mathematical concept involved in this process is the so-called *division algorithm*. We explain it in current notation (which has been around for only the last four or five centuries). When a number of objects is divided into groups of size d (in our example, $d = 5$) it results in a certain number q of groups and a certain number r of 'leftovers'. For example, if we have $d = 5$ we can write 17 as $3 \cdot 5 + 2$: three groups of five and two singles. In general, for any number n we can write n as $q \cdot d + r$ where q is called the *quotient* and r is called the *remainder*. Note that $0 \leq r < d$. Thus we have, with a slight change in notation, the following fact.

Theorem 1.(Division Algorithm) *Given integers n and β with $\beta \geq 1$, there are unique integers q and d with $0 \leq d < \beta$ such that*

$$n = q\beta + d. \tag{1}$$

Proof. Let $q\beta$ be the largest multiple of β such that $q\beta \leq n$. Put $d = n - q\beta$. This clearly gives (1) and q and d are unique and d is as desired. Q.E.D.

Repeated use of this process leads to the *digital representation* of a number n. The size β of grouping is called the *base*, the successive remainders the *digits*.

Theorem 2. *Let $\beta > 1$ be an integer. Then any integer $n \geq 1$ has the unique representation*

$$n = d_0 + d_1\beta + d_2\beta^2 + \quad + d_r\beta^r \tag{2}$$

with $0 \leq d_j < \beta$ *for all* j, $0 \leq j \leq r$, *and* $d_r \geq 1$.

Proof. By repeated use of the division algorithm we write

$$\begin{aligned} n &= q_1\beta + d_0 & 0 \leq d_0 < \beta \\ q_1 &= q_2\beta + d_1 & 0 \leq d_1 < \beta \\ &\vdots \\ q_{r-1} &= q_r\beta + d_{r-1} \end{aligned}$$

until we arrive at $q_r < \beta$.

Then

$$\begin{aligned} n &= q_1\beta + d_0 \\ &= (q_2\beta + d_1)\beta + d_0 \\ &\vdots \\ &= (. \quad ((q_r\beta + d_{r-1})\beta + d_{r-2})\beta + \quad + d_1)\beta + d_0 \\ &= q_r\beta^r + \quad + d_1\beta + d_0 \end{aligned}$$

which is (2) with $d_r = q_r$. Q.E.D.

The two theorems on p.10 were, in essence, known to the Egyptians, Babylonians, Chinese, Indians, and in the 'new' world to the Mayans and Aztecs. In Egypt, base two was used. Babylonians used a combination of base twelve and sixty. The Mayans and Aztecs used base twenty whereas the Indians and Chinese used base ten.

The representation (3) is the core idea behind our *decimal number system* which originated in India. The word decimal is of Indo-European origin. The word for tenth is *decimus* in Latin and *daśama* in Sanskrit. A shorthand way for writing (2) is

$$n = (d_r d_{r-1} \quad . d_0)_\beta$$

or simply

$$n = d_r d_{r-1} \quad . d_0 \qquad (3)$$

if the base β is understood (as ten for our decimal system). For example, 324 in base ten represents four, plus two tens, plus three tens of tens (i.e., hundred). We call d_j the j th *digit*, d_0 being the 0th digit. Actually, in a literal sense, the term digit (which means finger in Latin)

Around 750 BC the Indians developed the base ten place-value system we use today. However, it wasn't until the 1st century AD that they started using the symbol representing the digit zero as a number. The symbol 0 as a number is found in a manuscript called the Bakshali manuscript.

should be used exclusively when using our decimal base. The order in which digits appear holds special meaning. A digit takes on completely different meanings depending on its position. This is called the *place-value system.* For example, the digit 2 in 21 holds a different meaning than in 245. In the first, 2 represents two tens, while in the second 2 represents two tens of tens.

The two theorems stated above were, in essence, known to the Egyptians, Babylonians, Chinese, Indians, and in the 'new' world to the Mayans and Aztecs. In Egypt, base two was used. Babylonians used a combination of base twelve and sixty. The Mayans and Aztecs used base twenty whereas the Indians and Chinese used base ten. Around 750 BC the Indians developed the base ten place-value system we use today. However, it wasn't until the 1st century AD that they started using the symbol representing the digit zero as a number. The symbol 0 as a number is found in a manuscript called the Bakshali manuscript. It is a handbook on arithmetic and algebra written on birch bark in old Sanskrit. Discovered in 1881 in the village of the same name, the Bakshali manuscript is now housed in the Bodleian Library at Oxford University. Before showing how the representation (3) leads, naturally, to the most convenient algorithms for doing arithmetic, we shall first have a look at various systems of representing numbers.

Roman Numerals

It is evident that the Romans did not spend much time on developing a good system to represent numbers. The symbols they used for various numbers are illustrated in *Figure* 1. The symbols are then combined to express any number up to only a few thousand. To do so, it is recommended that the following rule be used: First write thousands, then hundreds, then tens, and lastly, units. Thus we write 1666 as MDCLXVI.

I	II	III	IV	V	VI	VII	VIII
One	Two	Three	Four	Five	Six	Seven	Eight
IX	X	XI	XII	L	C	D	M
Nine	Ten	Eleven	Twelve	50	100	500	1000

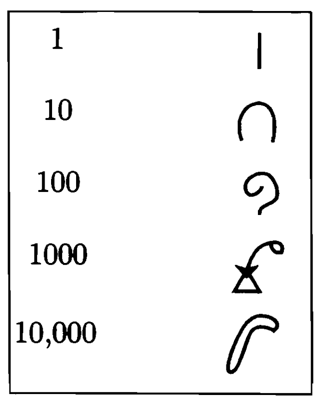
Figure 1. Table of Roman numerals representing various counts.

One difficulty with Roman numerals is that the representation of a number is not unique. For example, as watch makers know, four can be written as IIII or as IV. Representation can also be ambiguous if one does not know the convention; for example, the number XIX can be interpreted as X(IX), or (XI)X. There are rules for Roman numerals so that parentheses are not needed. If a larger numeral follows a smaller one, these two are taken together. For example, XIX is read as X(IX), not (XI)X. We leave it to the imagination of the reader to see the utility of the Roman numerals in arithmetic (addition, subtraction, multiplication and division).

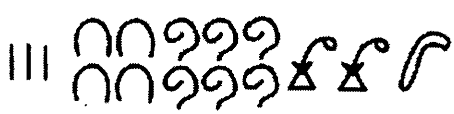
Egyptian Numerals

A somewhat more sophisticated example of grouping in number representation was developed by the Egyptians some 5000 years ago, long before the Romans. In this hieroglyphic system, each of the first several powers of ten is represented by a different symbol as illustrated in *Figure* 2.

Figure 2. Egyptian numerals.



Arbitrary whole numbers are represented by appropriate repetition of the symbols. Thus 12,643 can be written as



Note that unlike our modern place-value system, Egyptians wrote numbers in ascending order of magnitude, beginning with writing the units on the left. However, the order in which these symbols are arranged is immaterial because there is no place value in this system.

This way of writing numbers has one major drawback. One needs a symbol for each power β^j. Because j can be arbitrarily large, there is no end to the need for symbols. On the other hand, it was realized fairly early in India (around the middle of the first millennium BC) that because in representation (2), the successive digits d_j are all smaller than β, which is fixed, all we need are β symbols, one for each digit from zero to $\beta - 1$. Then to write n, it suffices to abbreviate (2) as $n = d_0 d_1 \quad . d_r$ (or $d_r \quad . d_1 d_0$), where the symbol d_j in the jth place would represent $d_j \beta^j$. This place value system of writing numbers, now used internationally as decimal system with ten digits, is undoubtedly unsurpassed and one of the most profound ideas in the history of mathematics. Because of its simplicity it may seem as natural as it could be, but we aim to show that this is not so.

Chinese Numerals

The Chinese invented the counting board – a table on which counting rods (small bamboo rods about 10 cm long) were manipulated to perform various calculations (see *Figure* 3). Note that the vertical arrangement of the number is listed first followed by the horizontal arrangement.

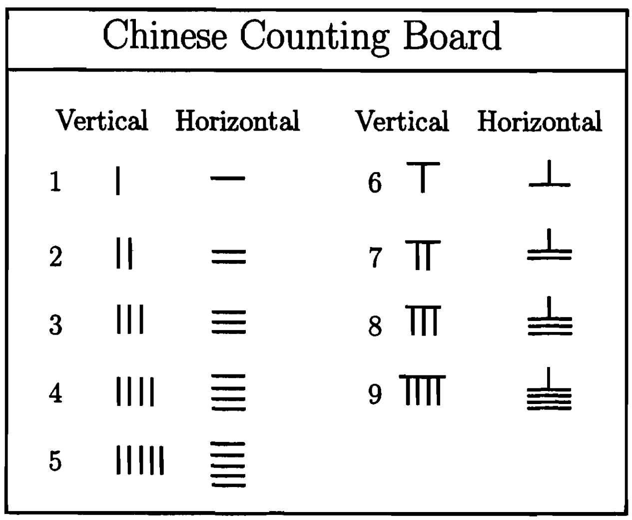
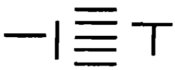
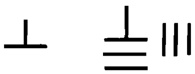


Figure 3. Arrangements of the counting rods for the numbers 1 through 9.

To represent numbers greater than ten the Chinese used a decimal place-value system on the counting board. The rods were set in columns, the rightmost column holding the units. To aid in reading the numbers, the vertical arrangement of a digit was used in the units column, followed by a horizontal arrangement of a digit in the tens column and so on, alternating vertical and horizontal arrangements. Thus 1156 would be represented as



and 6083 would be represented as



There was no symbol for zero. Note that a space is left between the two non-zero digits.

This system would have been confusing, for there is no way of telling if there is indeed a zero in this space, were it not for the alternating arrangement of symbols. The Chinese, seeing two horizontal arrangements in a row would immediately know that there was nothing in the hundreds place.

Differing processes were used to perform addition, subtraction, multiplication and division. Negative numbers could be distinguished from positive numbers by coloring. For example, one could color all positive numbers black and all negative numbers red. Manipulations on the counting board were eventually extended to such procedures as solving systems of linear equations and finding numerical solutions to polynomial equations.

Numerals of Native Americans

When the Europeans found the 'new world', the land was, according to them, inhabited by primitive and uncivilized people. However, the archaeological discoveries

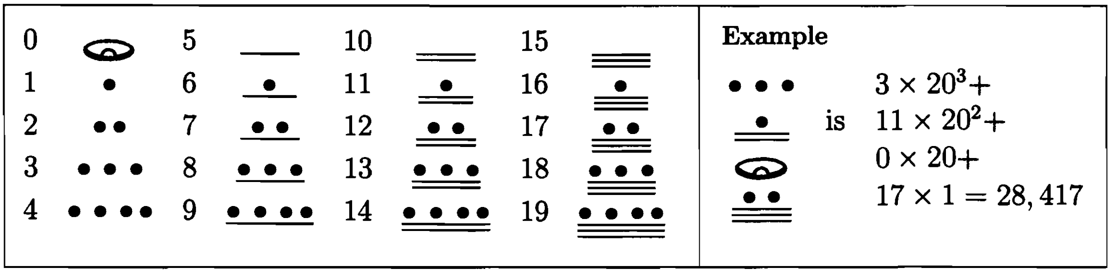


Figure 4. Mayan numerals.

in both South and North America tell a different story. During the 'Dark Ages', a description of Europe during approximately the period 500–1250 AD, the Americas had some of the most developed civilizations of the time. The most notable are the Mayan and the Inca empires. The Mayan civilization flourished on the Yucatan peninsula of Mexico. During this period, the Europeans were using the Roman numerals, which provide representation of numbers up to only a few thousand and their utility is limited to keeping records involving small numbers, such as that of important dates. On the other hand, the Native Americans had a fairly advanced mathematics for that period in the history of mankind (see [1]). The Mayans had a place value system similar to our current decimal number system. Theirs was a vigesimal system (base twenty). They also had a symbol for zero. Their symbols for zero and other digits are shown in *Figure* 4. In this place value system, the numbers were written from top to bottom. (see example in *Figure* 4)

The Mayans had a place value system similar to our current decimal number system. Theirs was a vigesimal system (base twenty). They also had a symbol for zero.

The Mayans were accomplished astronomers. Their year had 360 days, consisting of eighteen monthly periods of twenty days, and an extra month consisting of only five days. They could predict astronomical events, such as eclipses, for the next several hundred years, with an error of no more than one day. For details, see ([2], pp. 49-54).

The Incas of South America built a vast empire known as the Kingdom of the Four Directions. A peculiarity of

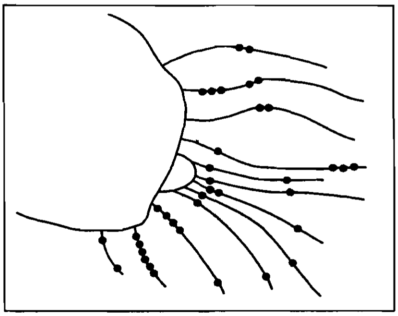


Figure 5. Quipu.

the Inca people is that they had no form of written language. Instead, the Incas communicated throughout the empire through a quipu. A *quipu* is a collection of colored knotted cotton strands (see *Figure* 5). The colors of the strands, their placement on the main strand, the spaces between the strands, and the type and placement of knots on each individual strand all played important parts in interpretation of the information.

The color of the strand represented the item that was being counted. For example, yellow would represent gold, red for the army, or gray for sheep. If one could not distinguish the objects through color, they were then distinguished through quality. In weaponry, the count of the lance would be first since it was considered the most honorable weapon by the Inca.

The system of counting on the strands was done in base ten. The units were knotted at the bottom of the strand whereas the hundreds or thousands would be knotted at the top of the strand. The spacing of the groups of knots had to be precise so that an absence of a group of knots could be noticed when a zero was needed.

A peculiarity of the Inca people is that they had no form of written language. Instead, the Incas communicated throughout the empire through a quipu. A *quipu* is a collection of colored knotted cotton strands.

Only .those trained in quipu could read the strings. The *Camayus* were trained in the art of reading and creating quipu. The calculations were performed on the Inca abacus, called *yupana* (see *Figure* 6), before they were recorded on a quipu. For details see ([2],pp.28-41). The quipucamyus would create quipu and submit them to the central government so that the ruling class would

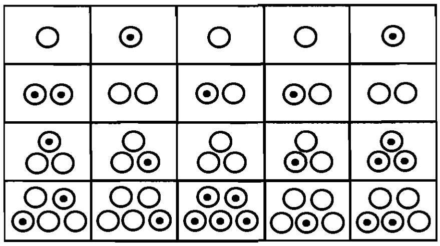


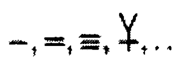
Figure 6. An Inca Yupana.

know the exact economic conditions of all regions of the empire. Thus the Incas could record their history, their laws, and contracts through the quipu without the use of a written language. We are not unfamiliar with this concept. In computing today, each character can be represented by numerals. In a sense, the Incas read a type of machine language common to computers today.

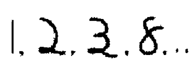
The Incas could record their history, their laws, and contracts through the quipu without the use of a written language. We are not unfamiliar with this concept. In computing today, each character can be represented by numerals. In a sense, the Incas read a type of machine language common to computers today.

Hindu–Arabic Numerals

Finally, we comment on our present and internationally adopted decimal number system. As said earlier, it originated in India. From there it spread to Arabia. The Italian mathematician, Fibonacci, who was a frequent visitor to the Islamic world, learned it from Islamic scholars and introduced it to Europe around 1220 AD through his famous book *Liber Abaci.* Soon thereafter, the European colonists, who conquered most of the world, are responsible for making it international. Our decimal numerals $1, 2, 3,$ started as



which when written hurriedly look like



Their evolution to the present form is illustrated in *Figure* 7.

The symbol 0 was invented a little later; first, to indicate an empty place in our place-value number system and then to denote the number zero. Fibonacci begins *Liber Abaci* by introducing these numerals:

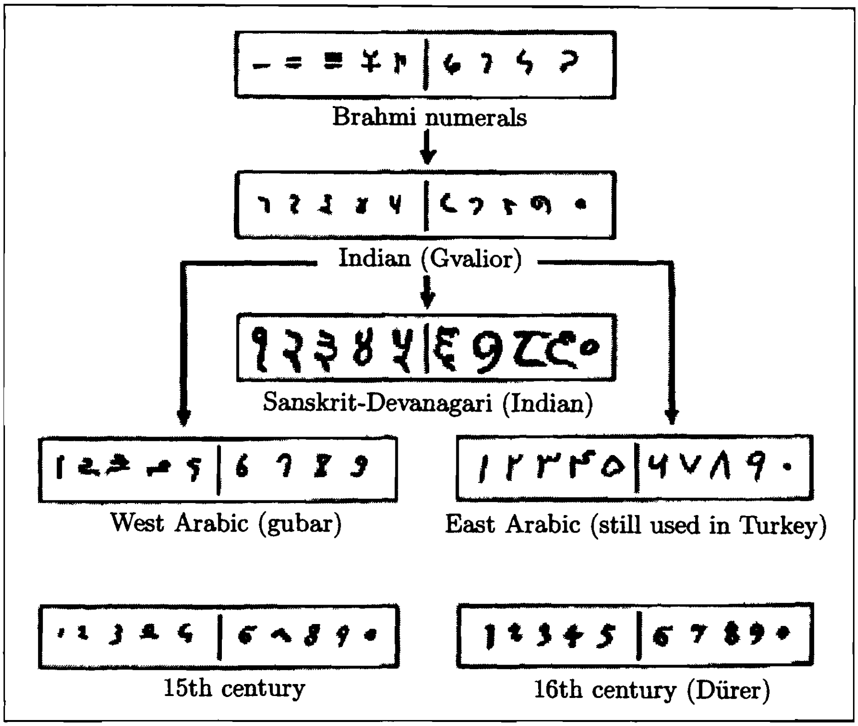


Figure 7. Development of our modern numerals.
(Source: From Number Words and Number Symbols, A Cultural History of Numbers by Karl Menninger. Translated by Paul Broneer from the revised German edition. English translation ©1969 by The Massachusetts Institute of Technology. Reprinted by permission.)

"The nine figures of the Indians are 9, 8, 7, 6, 5, 4, 3, 2 and 1. With these nine figures, and with the sign 0, which Arabs call zephirum (cipher) any number can be written as we shall show."

The importance of the Hindu–Arabic numerals to the world of mathematics cannot be overstated. According to the great French mathematician, Laplace

It is India that gave us the ingenious method of expressing all the numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value. It is a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it has lent to all computations puts out arithmetic in the first rank of useful inventions. We shall appreciate the grandeur of this achievement when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity.

Greek Numerals

The Greek numerals resembled the Roman numerals in concept, and thus were not based on place value system. One of the widespread systems used for the Greek numerals was the so called 'Attic' system that existed during the first millenium BCE. The symbols for

1 5 10 100 1000 and 10,000

were respectively

| 𐅃 Δ H X and M

Only the first symbol is a vertical bar representing one. The others are the first letters of the corresponding Greek words for the numbers:

5	𐅃	Pi (the earlier version of π)	Pente
10	Δ	Delta	Deka
100	H	Eta	Hekaton
1000	X	Khi	Khilioi
10000	M	Mu	Murioi

The numbers 50, 500, 5000 and 50,000 were made up of combinations of 5 (i.e., 𐅃) and the corresponding quotients:

50 = 𐅄 500 = 𐅅 5000 = 𐅆 50000 = 𐅇

Note that in the Roman system, M stands for one thousand and not ten thousand. The Attic system was mainly used by the Athenians.

The number 6789 would correspond to the long string of symbols:

𐅆X 𐅅HH 𐅄ΔΔΔ 𐅃||||

Differnt Greek states had different systems but similar in concept to one another. These systems together came to be known as 'Acrophonic' number systems. Some of them existed during 1500-1000 BCE also.

– Editors

Source: *The Universal History of Numbers*, Translated by David Bellos and E F Harding, Vol. I, pp. 355-365, Penguin Books India, 2005. The original is by Georges Irfah in French (1994).

Address for Correspondence
Jasbir S Chahal
Professor of Mathematics
Brigham Young University
USA
Email:jasbir@math.byu.edu

Suggested Reading

[1] **M P Closs (Ed.), *Native American Mathematics*, Univ. of Texas Press (1986).**

[2] **G G Joseph, *The Crest of the Peacock, Non-European Roots of Mathematics*, Princeton University Press (2000).**

[3] **K Menninger, *Number Words and Number Symbols*, Dover (1992).**