Schrödinger's Uncertainty Principle?

Lilies can be Painted

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The famous equation of quantum theory,

$$\Delta x \Delta p_x \geq h/4\pi = \hbar/2$$

is of course *Heisenberg's* uncertainty principle[1]! But Schrödinger's subsequent refinement, described in this article, deserves to be better known in the classroom.

Let us recall the basic algebraic steps in the textbook proof. We consider the wave function (which has a free real parameter α) $(\hat{x} + i\alpha\hat{p})\psi \equiv x\psi(x) + i\alpha(-i\hbar\partial\psi/\partial x) \equiv \phi(x)$, The *hat* sign over x and p reminds us that they are operators. We have dropped the suffix x on the momentum p but from now on, we are only looking at its x-component. Even though we know nothing about $\psi(x)$ except that it is an allowed wave function, we can be sure that $\int \phi^* \phi dx \geq 0$. In terms of ψ, this reads

$$\int \psi^*(\hat{x} - i\alpha\hat{p})(\hat{x} + i\alpha\hat{p})\psi dx \geq 0. \qquad (1)$$

Note the all important minus sign in the first bracket, coming from complex conjugation. The product of operators can be expanded and the result reads[2]

$$< \hat{x}^2 > + < \alpha^2 \hat{p}^2 > + i\alpha < (\hat{x}\hat{p} - \hat{p}\hat{x}) > \; \geq 0. \qquad (2)$$

The three terms are the averages of (i) the square of the coordinate, (ii) the square of the momentum, (iii) the "commutator" $\hat{x}\hat{p} - \hat{p}\hat{x}$. It was Heisenberg's insight in 1925 that this commutator equals $i\hbar$ which gave birth to quantum mechanics! We thus have a real quadratic expression in α which can never be negative. So the discriminant '$(b^2 - 4ac)$' of the quadratic is negative or zero. This gives,

$$< \hat{x}^2 >< \hat{p}^2 > \; \geq \hbar^2/4. \qquad (3)$$

Equation (3) looks like the uncertainty principle after taking the square root on both sides. But Heisenberg's Δx

[1] Δx is the uncertainty in the x component of position and Δp_x the uncertainty in the x component of the momentum of a particle.

[2] The angular brackets < > denote the quantum-mechanical average value.

is the root mean square deviation of x from its average, $\bar{x}$. Similarly, we need $p - \bar{p}$. But no more work is needed! The only property used in deriving (3) was the commutation relation $AB - BA = i\hbar$, and this is equally true if we take $A = \hat{x} - \bar{x}$, $B = \hat{p} - \bar{p}$. In physical terms, we are choosing the average x as the origin and the velocity of our frame of reference to be the average velocity.

So far so good. But Schrödinger's insight was that we have even more freedom in choosing A and B, keeping their commutator equal to $i\hbar$. For example, take $B = \hat{p}$ as before, but $A = \hat{x} + \beta\hat{p}$ with β a real parameter. We still have the same commutator, so $< A^2 >< B^2 > -\hbar^2/4 \geq 0$. Note again that the left hand side is quadratic in the free parameter β, and hence we can be sure the discriminant is zero or negative. This condition now gives (check it out!) the Schrödinger form of the uncertainty relation

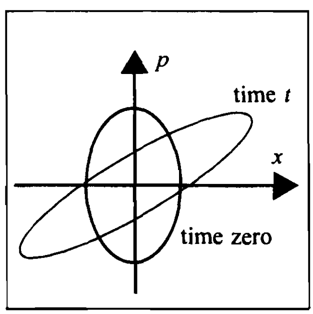
$$< \hat{x}^2 >< \hat{p}^2 > - < \frac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2} >^2 \geq \hbar^2/4. \quad (4)$$

Clearly, this is better than Heisenberg's, since the square of the (real!) average of $(\hat{x}\hat{p} + \hat{p}\hat{x})/2$ can be moved to the right hand side of (4), which is therefore greater than the right hand side of (3). Notice that there is no difficulty in measuring x and p from their average values. If we do so, equation (4) reads.

$$< \Delta\hat{x}^2 >< \Delta\hat{p}^2 > - < \frac{\Delta\hat{x}\Delta\hat{p} + \Delta\hat{p}\Delta\hat{x}}{2} >^2 \geq \hbar^2/4. \quad (5)$$

What does it all mean? We now reveal what β is by putting it equal to t/m. $x + pt/m$ is just the position at a time t, assuming a free particle of mass m. We thus need to understand how an uncertainty product like $\Delta x \Delta p$ behaves under free motion. Let us first look at a classical situation in which we have a cloud of particles, occupying an elliptical region in $x - p$ space (also known as phase space). For simplicity, we have chosen the mean value of x and that of p to be zero (see figure). At a later time, the particles with positive p have moved to the right and those with negative p to the left. Our ellipse is still an ellipse but has got tilted, preserving its area[3]. Notice that the spread in p has remained the same, but the spread in x has increased. The Heisenberg uncertainty product $\Delta x \Delta p$ would thus increase. But

[3] This is in fact Liouville's theorem of classical mechanics about motion in phase space.



interestingly, the Schrödinger version in (5) retains the same value at later times, being simply related to the square of the area of the ellipse, which is fixed. The increase in $< \Delta\hat{x}^2 >$ is compensated by the build up of the term $< (\hat{x}\hat{p}+\hat{p}\hat{x})/2 >^2$.

This term was initially zero because of cancellation among the four quadrants. But after a time t, we see that the ellipse preferentially fills the first and third quadrants. We say that a correlation has built up between x and p. A statistician would say that x and p are no longer independent. For example, it is clear from the figure that the unconditional probability that p is positive is half. But if x is given to be positive, the probability that p is positive is greater than 1/2!

The reader who has not already noticed should be warned that after equation(5), the discussion is only meant to be plausible. Should one even be talking of phase space in quantum mechanics? Fortunately, many years after Heisenberg and Schrödinger, Wigner found a correct way to use such phase space pictures in quantum theory. For free particles and harmonic oscillators, the time dependence of this phase space distribution invented by Wigner is correctly given by classical mechanics, even though the wave function obeys the Schrödinger equation. For example, the spreading out of Δx which we inferred from our classical picture is a well-known phenomenon called 'wave-packet dispersion'. Incidentally, in classical theory it would have been sufficient to use the average $< xp >$ to reveal this correlation between x and p. In quantum theory $< \hat{x}\hat{p} >$ is not even real, but $< \hat{x}\hat{p} + \hat{p}\hat{x} > /2$ is real and is the correct way to quantify correlation between x and p. The virtue of Schrödinger's version (5) is that it accounts for this correlation. In special cases like the free particle and the harmonic oscillator, the 'Schrödinger uncertainty product' even remains constant with time, whereas Heisenberg's does not.

The glory of giving the uncertainty principle to the world belongs to Heisenberg. But we see that Schrödinger was able to see further, standing on his shoulders. These ideas were carried even further in the field of quantum optics, but that is another story.

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