The Chandrasekhar Limit

Stars are stable against collapse because internal pressures balance gravity. In gaseous stars like the Sun the internal pressure is due to the *thermal motion* of the atomic nuclei and the electrons, and also the pressure of the radiation generated by the thermonuclear fusion reactions. After many billions of years in the case of stars like the Sun or a few million years in the case of more massive stars, the fusion reactions will cease and the stars will at last find peace. The three possible end states of stars are *white dwarfs, neutron stars* and *black holes*.

Let us initially focus our discussion on *white dwarfs*. Since there is no energy generation in these stellar remnants, the key question is the following: How do they support themselves against gravitational collapse? After all, as a white dwarf radiates away the fossil heat the thermal pressure will decrease and eventually vanish. Such a star would thus appear to be doomed. And yet, white dwarfs live for ever!

The resolution of this major paradox was given by R H Fowler in 1926. The observational clue was that the mean density of white dwarf stars is of the order of 10^6 g cm^{-3} (in comparison, the mean density of the Sun is roughly 1 g cm^{-3}). Fowler argued that at such high densities the pressure due to the electrons should be calculated according to the rules of the quantum statistics, which had just been discovered by Fermi and Dirac. It is *not* attributed to their *thermal* motions. According to the Fermi-Dirac statistics, a gas of electrons will have finite energy even at the absolute zero of temperature. This arises due to the *Pauli exclusion principle* according to which not more than two electrons can occupy a given energy level. The energy content will therefore increase dramatically as the density of the electrons increases. Indeed, one cannot pack electrons into a shrinking volume without correspondingly increasing their kinetic energy. Hence a white dwarf will contract to a sufficiently high density such that the pressure coming from zero point motion will balance gravity. This was Fowler's resolution of the paradox.

The above argument implies that all stars, however massive, can find ultimate peace as white dwarfs. This may be seen as follows. The quantum mechanical pressure of an electron gas (equal to (2/3) × energy per unit volume) is found to be proportional to the 5/3 power of its number density n; equivalently if ρ is its mass density $P_e = K_1 \rho^{5/3}$. The mechanical stability of a star is governed by the condition of *hydrostatic equilibrium* of each layer. The inward force due to the gravity of the mass $M(r)$ within the spherical layer balances the outward force exerted by the pressure difference across it. In symbols

$$\frac{dP_e}{dr} = -\frac{GM(r)\,\rho(r)}{r^2}.$$

Combining this with the earlier expression for the pressure, one can show that the *radius of a white dwarf will be inversely proportional to the cube root of its mass*. Thus a white dwarf of any mass will come to equilibrium at a finite radius.

The above relation $M \propto R^{-3}$ implies that as the mass of the white dwarf increases, its mean density will increase as the *square* of the mass. Let us now recall that as the density increases the kinetic energy of the electrons, or more basically, the momentum of the electrons will increase, and consequently special relativistic effects will become more and more important. This has an important bearing on the dependence of the pressure on density. The mass-radius relation derived from a more exact theory including special relativity for the electrons differs dramatically from the relation described earlier. Not surprisingly, the approximate (nonrelativistic) theory is excellent for sufficiently low mass white dwarfs (say, less than half a solar mass). But as the mass increases the exact theory predicts *much smaller radii*, till eventually at a critical mass the radius goes to zero! This critical mass is known as the *Chandrasekhar Limit* . Stars more massive than this simply cannot be supported against gravity by the quantum mechanical pressure of electrons. The value of this limiting mass was derived by Chandrasekhar in 1930 using the following simple argument.

Because of the gradient of density in a star, one would expect special relativistic effects to first become important near the centre of a white dwarf. As we go to more and more massive white dwarfs, special relativity will dominate in a larger and larger mass fraction of the star till, eventually, one will reach a mass for which a relativistic description is essential for the entire star. In this limit one may assume the electrons to be ultrarelativistic (namely, their rest mass energy can be neglected compared to their kinetic energy). In this limit the pressure of the electrons is given by $P_e = K_2 \rho^{4/3}$ (instead of $\rho^{5/3}$). The equation of hydrostatic equilibrium admits an exact solution for the mass given this particular form for the density dependence of the pressure:

$$M = 0.197 \left[\left(\frac{h\,c}{G} \right)^{3/2} \cdot \frac{1}{m_H^2} \right] \frac{1}{\mu_e^2} .$$

Here, h is Planck's constant, c the velocity of light, G the Newtonian constant of gravity, and m_H is the mass of the hydrogen atom. μ_e is the mean molecular weight per electron (since one can safely assume that all the hydrogen would have been consumed in a white dwarf, $\mu_e \approx 2$). For $\mu_e = 2$, the numerical value of the above mass is ≈ 1.4 times the solar mass. Even before he did the exact theory Chandrasekhar correctly identified this to be the *limiting mass* of ideal white dwarfs. It is interesting that fifty years later, the masses of many *neutron stars* have been measured by astronomers to be very close to 1.4 times that of the sun! Our current understanding is that at an earlier stage, this matter was at the centre of a massive star and collapsed as it passed the Chandrasekhar limit, accompanied by an explosive release of energy which ejected the remaining mass as a supernova. In his own lectures, Chandrasekhar was fond of drawing a parallel with the situation of a nucleus with charge $+Ze$ and a single electron of charge $-e$. The non-relativistic quantum theory predicts that the electron cloud has a size proportional to $(1/Z)$. But in the relativistic theory, it is found that there is a critical value of $Z \approx 137$, beyond which the electron "collapses" to zero radius, at least when we model the nucleus as a point charge! Relativity and quantum theory, the two great revolutions of twentieth century physics, combine to set the scale for the maximum size of both an atom and a stellar remnant.

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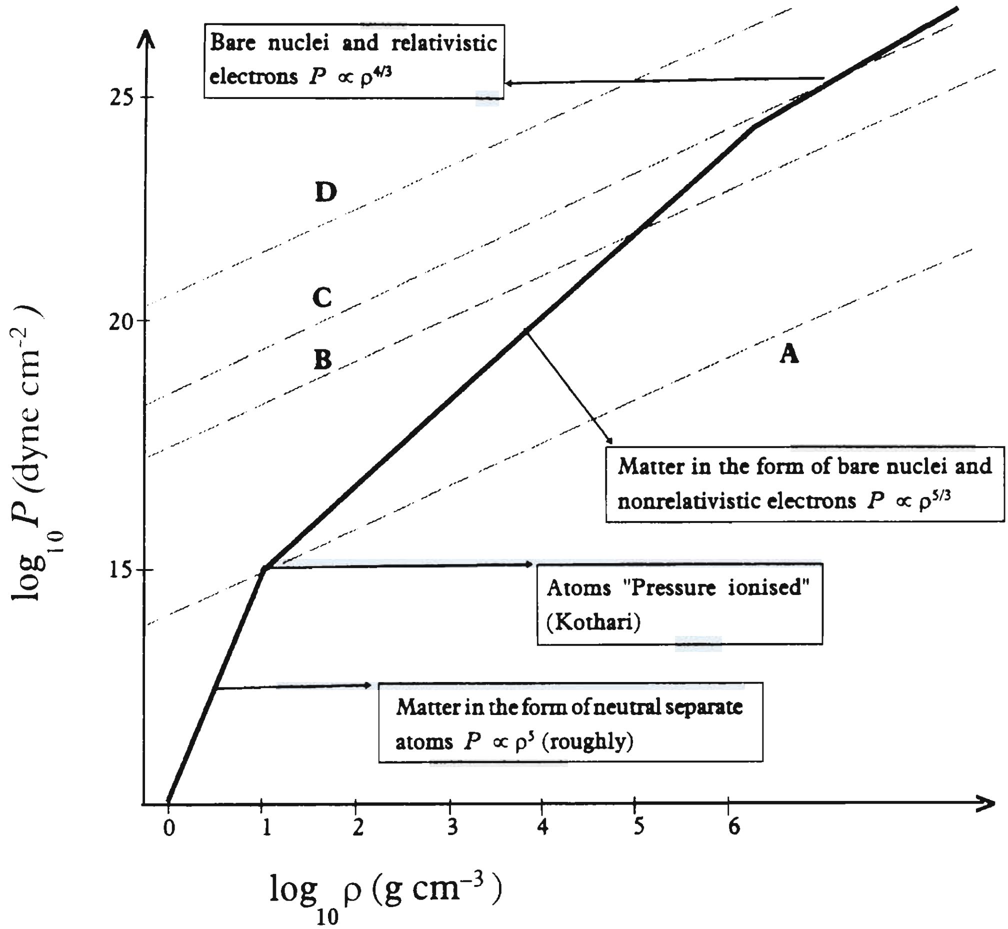

The solid line shows schematically the pressure-density relationship for cold matter. The dashed lines show the pressure P_g needed to balance gravity in a body of mass M and radius R as a function of its density ρ. Intersection with the solid line is the condition for equilibrium $P_g \propto (\frac{GM^2}{R^2}) / R^2 = \frac{GM^2}{R^4}$. Since $\rho \propto M/R^3$, $P_g \propto \rho^{4/3}$.

P_g vs ρ is shown for four cases

(A) Jupiter ; (B) A white dwarf of mass < 1.4 solar masses; (C) A white dwarf of mass nearly 1.4 solar masses. Notice that the equilibrium occurs at a very high density, i.e. very small radius; (D) A cold body of mass >1.4 solar masses. No equilibrium is possible at any radius.

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