The Story of Nuclear Matter

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R Rajaraman started his research work in the area of nuclear many-body theory. Subsequently he enlarged his interests to include particle physics phenomenology, quantum field theory, soliton theory and quantum Hall systems. In recent years he has also been working on technical and policy aspects of reducing nuclear weapon dangers in South Asia.

This article gives an introduction to the Nuclear Matter – a subject to which Professor Bethe had made seminal contributions. Most of the discussion here will be devoted to describing the ideas that led up to postulating the nuclear matter system. That will be followed up with a qualitative description of the difficulties faced and the methods developed in tackling this system from first principles. The article will also be used as an excuse to explain many basic features of nuclear physics to students, using simple arguments.

Introduction

Professor Hans Bethe was one of the founding fathers of nuclear theory. Starting from the early thirties, he made numerous pioneering contributions to different aspects not only of nuclear physics, but practically every branch of physics[1]. Finally, in the mid-fifties, he embarked on the task of solving the Nuclear Matter problem. It took over a dozen years before he and his co-workers completed this task. Of course Bethe was doing many other things alongside, both in physics and on serious public policy issues. But the nuclear matter problem was perhaps his longest project in physics. This was also the period when the author had the invaluable opportunity of working closely with Professor Bethe. It is therefore tempting to say something about this subject here.

[1] Some of these contributions have been discussed in other articles in this issue.

But nuclear matter happens to be a particularly nasty and intractable many-body system, for reasons we will explain. Correspondingly, the theoretical structure put together for understanding it also turned out to be complex and elaborate. Therefore, rather than go into the intricacies of nuclear matter theory it may be more use-

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ful here to bring out the basic ideas in nuclear physics that lead up to and motivated the study of nuclear matter.

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What is nuclear matter? What is its significance to nuclear theory in general? What was so important about this system that Bethe, after decades of tackling so many different aspects of nuclei, decided to make its theory his last and longest project in nuclear physics? And what makes it such a difficult system to analyze? We will try to answer some of these questions here in simple terms and in the process also, hopefully, explain to student readers some basic concepts in nuclear physics. In its heydays in the mid-twentieth century nuclear physics was considered in the popular lexicon to represent the most rarified and abstruse of sciences. Yet it is possible to extract many features of nuclear physics using exceedingly simple semi-classical arguments, bordering on just commonsense.

Let us begin with a basic question whose answer is often, *post facto*, taken for granted. Recall that about a century ago, as the frontiers of physics took us from macroscopic objects down to atoms and molecules, the physics changed in a dramatic way to the extent that even the philosophical foundations of science had to be modified in a fundamental sense. The time-tested ideas of deterministic classical physics had to be replaced by a probabilistic, far less intuitive, quantum picture. These dramatic changes happened when we went from, say, grains of sand, for which classical physics is still valid to typical molecules which are smaller in size by a factor of 10^{-6} (a millionth). In the same way, as nuclei and their properties were being discovered, physicists could well have wondered if the same laws of quantum physics which work so spectacularly for atoms and molecules would continue to work equally well for nuclei. After all, as you go down from atoms to nuclei, you are again going down in size by a huge factor of about 100,000 from Angstroms (10^{-8} cm) typifying atomic distances, to Fermis (10^{-13} cm), the scale of nuclear sizes. Would the basic laws of quantum physics survive this transition?

As it happened, when calculations were done, one after another, in the nascent field of nuclear physics using the same principles of quantum mechanics that worked for atoms, those principles seemed to hold equally well here too. It helped that nucleons (the common name given to both protons and neutrons) in nuclei are also, to a fair approximation, non relativistic, although less so than electrons in atoms[2].

[2] The electron in the H atom, for instance, moves, on the average, at about one hundredth of the speed of light ($v/c \approx 1/100$) whereas for nucleons $v/c \approx 1/10$. But relativistic effects depend only on the square of this fraction, i.e. on $(v/c)^2$ Therefore up to an accuracy of a few percent, non relativistic quantum mechanics can still be used for nuclei.

But while general principles of non-relativistic quantum theory seemed to hold in the nuclear realm as well, most calculations of nuclear structure, except for the deuteron and other few nucleon systems, were 'phenomenological' They did not start from *first principles*, by solving the basic Schrödinger equation for the nucleons inside the nucleus in terms of their primary forces. Instead they began at some intermediate level 'model' with names such as the Shell model, the Liquid Drop model, the Collective models and so on.

The absence of first principles calculations should not surprise us. A typical nucleus, say, that of aluminum, which is not an especially large one, has 12 protons and 14 neutrons. But we know that even in classical Newtonian mechanics it is hard to exactly solve problems involving more than 2 interacting particles. Every particle exerts a force on every other and the problem becomes highly coupled. When only two bodies are involved, such as the Sun and the Earth (the Kepler problem) one can 'separate' the problem into two uncoupled parts by changing to center of mass (c.m.) and relative coordinates. The c.m. travels with uniform velocity and one then needs only to solve the effective one-body problem involving the relative coordinate. The moment you have three or more bodies such separation is in general not possible. The c.m. coordinate can still be separated from the others but the rest remain entangled with one another forcing us to look for difficult approximations.

Individual electrons can be viewed as simply moving in the electric field of the nucleus, which can be taken to be at rest at the origin.

The same is true in quantum mechanics. We can analyse the hydrogen atom problem relatively easily since it involves only a proton and an electron. But when you go to larger atoms involving several electrons, the problem becomes impossible to tackle analytically even though the forces between the different particles obey the simple Coulomb's law. Various approximations, like perturbation theory, the variational method, the Thomas Fermi model, etc., need to be employed.

Diffculties of Nuclear Theory

The corresponding problem in nuclear physics, of several nucleons interacting quantum mechanically with each other, is even more difficult than the one involving several electrons in atoms. There are two reasons for this:

1. In an atom, there is great asymmetry between the status of the nucleus and that of the electrons. The nucleus is not only much heavier than the electrons, but also carries a large electric charge which acts like a strong point charge at the center. This can be exploited to set up approximations where the individual electrons can be viewed, in the first approximation, as simply moving in the electric field of the nucleus, which can be taken to be at rest (because of its relative heaviness) at the origin. Of course the electrons are also interacting with one other, but those effects are treated as corrections. This approach has had considerable success in atomic physics.

By contrast, in a nucleus all the nucleons are on an equal footing. They all carry roughly the same mass and exert similar forces on one another, and there is no natural center to which you can anchor your approximation process.

The nuclear forces holding nucleons together are much more complex than the electromagnetic force operating in atomic phenomena.

2. The nuclear forces holding nucleons together are much more complex than the electromagnetic force operating in atomic phenomena. The force between two nucleons (the so-called Strong force) , obtained by studying the scattering of nucleons off one another, consists of several terms whose dependence on distance is a complicated function, unlike the simple $1/r^2$ Coulomb force. Furthermore there is also the possibility of *intrinsic* three-body and higher many-body forces acting among nucleons. By intrinsic three-body forces we mean the following. Suppose we have three nucleons labeled 1,2 and 3 near one another. Then they will of course be acted upon by the three pair-wise potentials $v(r_{12})$, $v(r_{23})$, and $v(r_{13})$. But we may also have an additional potential $W_{123}(r_{12}, r_{23}, r_{13})$, which is not present when only two nucleons are around. Similarly, higher n-body forces are defined. Such intrinsic many-body forces are not present to any significant degree in our day-to day macroscopic world, nor even in atomic physics. But they were not invented by nuclear matter theorists just to complicate their lives. Quantum field theory which underlies Yukawa's famous explanation that nuclear forces are caused by the exchange of mesons, also predicts the existence of strong intrinsic n-body forces.

Before one can check whether such n-body forces are indeed present in real nuclei and theoretically calculate their effects, one must first learn how to tackle the nuclear many-body problem in terms of just the normal two-nucleon forces. That is hard enough to do, given the nature of the nucleon-nucleon (N-N) force.

Nucleon–Nucleon Potential

What is the form of the N-N force? Its details are deduced from two experimental sources. One is the deuteron. Since it is made of one proton and one neutron, there is no Coulomb force and so it is bound by just the nuclear force. Also, since it is a two-particle system, it is theoretically tractable and some properties of the nuclear force can be inferred from it. [For instance, experimental observation of the deuteron nucleus shows that its ground state, unlike that of the hydrogen atom, is not a purely spherically symmetric zero-angular momentum S-state, but instead has a small admixture of the D-state. This implies that the force is not entirely central; it turns out to have a 'tensor force' component which depends on the spins.] A second and much richer experimental source of information is nuclear reaction data, where nucleons and nuclei scatter from one another.

The very fact that nuclei exist tells us that there must be a new type of force acting between nucleons, different from other familiar forces.

But, even without going into detailed analysis of scattering data we can deduce some essential features of nuclear forces from very general properties of nuclei. Let us gather together these features in easy steps.

1. *Need for a New Nuclear Force*

Clearly the protons in the nucleus must repel each other due to their Coulomb force. So there must be some attractive force to overcome this repulsion, and also to hold neutrons together. It is easy to check, inserting the masses of nucleons into Newton's Law of Gravitation, that the force of gravity between nucleons is far too weak to overcome the Coulomb repulsion between protons. Thus the very fact that nuclei exist tells us that there must be a new type of force acting between nucleons, different from other familiar forces.

2. *A Strong Force*

This attractive nuclear force must be very strong in order to overcome the P-P Coulomb repulsion. We know that electrons in their atomic orbits are bound to the nucleus by energies in the tens to hundreds of electron-volt (eV) range. The Coulomb energy between protons in a nucleus will be 100,000 time stronger since their distances from each other are that much closer. Hence, to overcome such repulsion, the attractive nuclear force must have a potential energy in the MeV (millions of eV) range.

3. *Short Range*

We know that there are less than hundred different elements in nature. The heaviest of them (corresponding to atoms shown at the tail end of the Periodic Table), containing in turn the largest nuclei, have about 260 nucleons in them. So, despite this very strong attractive force that nucleons exert on one another, nuclei don't seem to grow to be arbitrarily large. (Contrast this with gravitational attraction which holds together objects of astronomical size). From this we can infer another feature of the nucleon-nucleon force. This force must have a very *short range*. It must fall off very steeply beyond a couple of Fermis, so that any given nucleon cannot help bind any other nucleon beyond this short range of a few Fermis. The repulsive Coulomb force between protons also falls with distance, but comparatively slowly as $(1/r^2)$, and hence can overcome the attractive nuclear force beyond this range[3]. This prevents nuclei from growing larger than a few Fermis in size.

[3] In the case of neutrons there is no Coulomb repulsion, but readers familiar with quantum effects will know that both the uncertainty principle and the Pauli principle effectively want to keep neutrons from coming too close. Therefore as the nuclear attraction rapidly weakens it is unable to hold even neutrons together.

4. *Repulsive Core*

Even though the short range of the nuclear attraction prevents nuclei from growing large in volume, why can't many more nucleons, especially neutrons, squeeze within this range? After all, if you add more and more nucleons into the same volume, the system should become more and more stable since each nucleon would now be held in by the combined attraction of all the others. Thus we should have nucleons of atomic weight in the thousands. That we don't have such heavy nuclei in nature is because the nuclear force also has a steep repulsive component at very short distances preventing us from packing them too close to each other. This short range repulsion extends up to about 0.5 to 0.7 Fermis depending on details. For simplicity it is often approximated by a 'hard core'. Outside the hard repulsive core lies the zone of strong attraction referred to above, which also tapers off exponentially fast outside its short range.

A qualitative sketch of the nucleon-nucleon potential can thus be drawn as in *Figure* 1. The actual nucleon-nucleon potential, obtained from detailed analysis of nuclear scattering data, is much more complicated, containing a sum of over a dozen different terms, depending on the spin, isospin and so on. But its general characteristics are as shown in the figure.

In short, we have been able to deduce just from well known and rudimentary properties of nuclei that, at the simplest level, one can picture nucleons as hard balls (because of the hard core) of radius about 0.5 Fermi which attract each other by the short ranged strong force. In addition of course we have the Coulomb repulsion between protons.

Let us see how much further we can go with just simple ideas. Given a nuclear force as shown, let us see what would happen if we start building larger nuclei beginning with smallest ones, like the deuteron. As we add more nucleons, each nucleon would be bound by attraction of the other nucleons. At the same time the nucleons will not squash into each other more than a certain amount

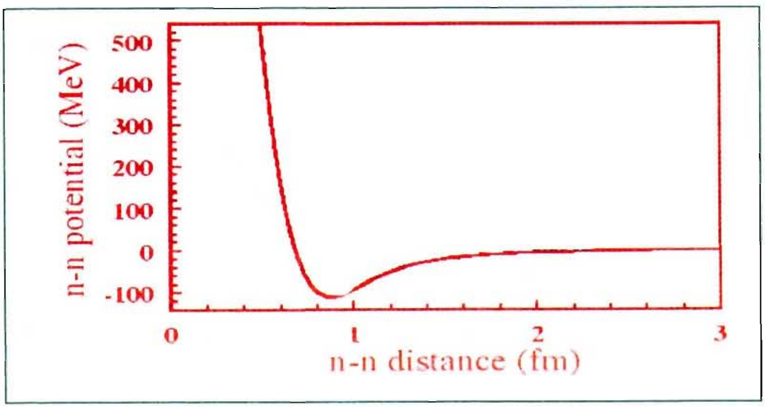

Figure 1. A schematic sketch of the N-N potential. This figure is extracted from the article by Jacek Dobaczewski, 2003-01-27, on the website http://www.fuw.edu.pl/ \~{}dobaczew/maub-42w/ node16.html

because of their repulsive cores. They would behave like little balls with a core radius, held together by the glue of their mutual attraction. Thus as we add nucleons the nucleus should tend to become larger retaining roughly the same density. This is what happens in reality. The radii of nuclei can be measured by scattering electrons off them. It is found, for instance, that while the radius of the oxygen nucleus (with 16 nucleons) is about 3 Fermi, that of lead (with 206 nucleons) is about 7 Fermi. If you compare the ratio of their volumes $[(3/7)^3 = 0.0787]$, to the ratio of their masses [16/206 = 0.0776] they are almost the same! Thus as you add more and more nucleons the nuclear size grows correspondingly, retaining the same rough density[4].

[4] This is quite different from the way atomic sizes change as you move along the Periodic Table. If you go along a given Period, i.e. move horizontally to the right on the Table, the heavier atoms with larger atomic number in fact become smaller in size. Their nuclei have larger electric charge and pull the electrons inwards making the same orbits smaller. Thus while lithium (Z=3) has a radius of about 1.52 Angstrom, fluorine (Z=9) has a size of only 0.72 Angstroms.

Binding Energies

Turning from sizes to energies, remember that that potential energy of the nucleus will be negative because it is bound. This corresponds, as per Einstein's famous $E = mc^2$ relation, to a reduction in the mass of the nucleus as compared to the sum of the masses of its constituent nucleons. The binding energy (B) per nucleon of the nucleus (the magnitude of the negative potential energy) is then given by

$$\frac{B}{A} = \frac{1}{A}\left(\sum_{1}^{A} m_i - M_A\right) c^2$$

where M_A is the mass of the full nucleus with A nucleons and m_i is the mass of its i th constituent nucleon. The masses of different nuclei are obtained from the masses of the corresponding atoms minus the masses of the electrons in them. Thus B/A is experimentally an easily available quantity. Now, if you were to begin with the deuteron and start adding nucleons, *each* nucleon will become even more strongly bound because it would feel the attraction of all the other nucleons. In more precise terms, the binding energy *per nucleon* will increase as

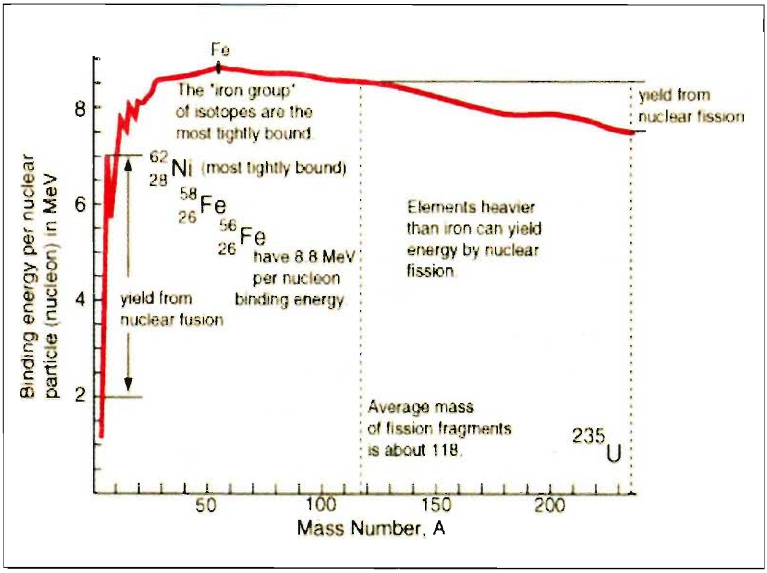

Figure 2. The binding energy per nucleon plotted as a function of the nucleon number A. This data is taken from the website. http://230nsc1.phy-astr.gsu.edu/hbase/nucene/nucbin.html

you add more nucleons and build bigger nuclei[5]. Eventually however, as you continue to build larger nuclei, a stage is reached when the outer nucleons no longer feel the pull of the innermost ones and vice versa, because they are outside the short range of each other's attractive nuclear force. After that the binding energy per nucleon no longer increases, and indeed should decrease a bit because the Coulomb repulsion between protons, with its long range, lowers the binding.

[5] When we add protons, their mutual Coulomb repulsion will offset their nuclear attraction but the latter will still win for small nuclei. We cant however, just to save Coulomb repulsion, build nuclei just out of neutrons, since the Pauli Exclusion Principle will resist bringing them too close to each other. Neutrons will have to be interspersed judiciously with protons. In nature small nuclei have roughly equal numbers of protons and neutrons.

These conclusions are borne out when we look at the observed (B/A) of different nuclei, given in *Figure* 2. We can see that the binding energy per nucleon does increase steeply as we go up from the deuteron, then levels off around a mass number of about 56 (Iron), and finally slowly decreases. For large nuclei it is in the neighborhood of 8.0 ± 0.5 MeV.

Of course we have used extremely simple arguments, and so our conclusions are very approximate statements. In reality the detailed variations as you go from one nucleus to the next are much more complicated, particularly for small nuclei, as the figure shows . For instance, as the figure shows the He nucleus ($A = 4$; also called the alpha

Fusion is hard to achieve because nuclei will be positively charged and will repel.

particle) is especially strongly bound compared to its neighbors. Calculating these specifics more precisely is the very complicated quantum many body problem of nuclear theory.

Fission and Fusion

One can also immediately understand from this binding energy graph the reason behind the famous phenomena of nuclear fission and fusion. If you start with a nucleus with very large A, on the right hand edge of *Figure* 2, its binding energy. per particle is lower than that of a nucleus in the middle regions of A. It is therefore energetically profitable for it to break up into smaller pieces. Radioactive nuclei do this spontaneously. But you can induce such break-up (fission) with a little help form the outside, i.e. by gently hitting the big nucleus with a neutron. This is what is done in reactors where uranium is bombarded with neutrons. The uranium (the isotope U^{235} to be precise) breaks up into daughter nuclei and emits the extra energy released. In the process more neutrons also are peeled off. These in turn can go and induce the fission of another uranium nucleus, and if this process continues one gets a chain reaction and massive amounts of energy released which can be converted to electric power (and alas, also to nuclear bombs).

Similarly, if you start with a couple of nuclei of very low A, one can again see that they have much lower binding energy per nucleon. It would again be profitable (i.e. exothermic) for them to combine ('fuse') into some larger nucleus which has more binding energy. This is the phenomenon of nuclear fusion. Unlike the case of fission, where all you need to do is hit a suitable large nucleus with a neutron, in the case of fusion, the two starting nuclei have to come close to each other in order to fuse. This is very hard to do because both nuclei will be positively charged and will repel at distances outside their nuclear force range. They both need to be very energetic to start with, in order to overcome their Coulomb repulsion and come close enough for the nuclear attraction to take over and make them fuse. This happens in the Sun and all the stars because their high temperature gives the starting nuclei enough kinetic energy. (One of the major discoveries of Bethe, listed in his Nobel citation, dealt with the precise fusion reactions taking place in stars). To do this in the laboratory in a controlled fashion has proved to be very hard and is still in the process of being achieved. When that happens it would be the first step towards the dream of fusion reactors.

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Semi-Empirical Mass Formula

You can check that the binding energy curve in *Figure* 2 can be fitted reasonably well by what is known as the Bethe–Weiszacker Semi-empirical Mass Formula. Writing the binding energy as $B = -E$ where E is the total energy of the nucleus, the formula is

$$E(A) = -a_\nu A + a_z A^{2/3} + \frac{a_e Z^2}{A^{1/3}} + \frac{a_e D^2}{4A},$$

where A is the nucleon number, Z is the atomic number, $D = A - 2Z$ and the other constants have values, $a_v =$ 15.78 MeV, $a_S =$ 16.060 MeV, $a_c =$ 0.6876 MeV, $a_e =$ 22.409 MeV.

(One can make further improvements to this formula to better fit the data in *Figure* 2, but we don't need to go that level of detail here.)

The importance of this very insightful formula is the physical significance that can be attributed to each term. Consider a particular nucleon inside a very large nucleus (large A). It will feel the attraction of only those neighbors who are within range of the nuclear force. As you keep increasing A further, the additional nucleons will not provide any further attraction. So this contribution to the energy *per nucleon* will saturate at some negative value independent of A for large A and the correspond-

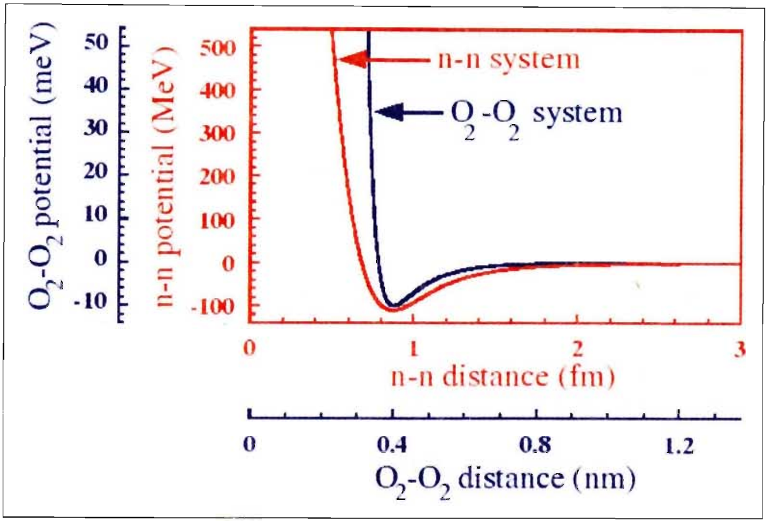

Figure 3. A comparison between inter nuclear forces (red lines) and inter molecular forces (blue lines). Figure extracted from Jacek Dobaczewski' article (see Figure 1)

ing total energy of the full nucleus will be proportional to A. This is the first term in the B–W mass formula and is called the volume term.

The second term is called the surface term and arises for the following reason. The nucleons at the surface of the nucleus have neighbors only on the inner side unlike nucleons in the interior. For them, the nuclear attraction will be less and first term is an overestimate. So the second term which corrects for this has the opposite sign. Also, in our simple classical picture of the nucleus as a ball made of hard sphere nucleons stuck to each other by nuclear attraction, the number of surface nucleons will be proportional to the surface area which in turn will be proportional to $A^{2/3}$.

The reader would have noticed that these two terms are similar to the bulk energy and surface tension energy of a liquid. In fact, the resemblance between the inter-nucleon forces and inter-atomic forces in liquids is very striking. In *Figure* 3 we show on the same graph the N-N force of *Figure* 1 and the inter-molecular forces in typical liquids. Of course, atomic (Van der Waals) forces are residues of electromagnetic forces between their electrons and protons. They are entirely different in strength (typically a few eV) and in range (a few Angstroms) from nuclear forces. But as *Figure* 3 shows, when suitably rescaled, the two are very similar. That is why a large nucleus can at a semi-classical approximation be viewed as a liquid drop. [This means that in principle nuclear matter methods can also be applied to a dense collection of atoms, such as liquids. Of course ordinary liquids in their ground state will not remain as liquids. Liquid He^3 is a possible candidate.]

A large nucleus can at a semi-classical approximation be viewed as a liquid drop.

Returning to the Bethe–Weiszacker formula, the first two terms discussed above correspond only to the nuclear force. The third term represents the energy resulting from the Coulomb force between the protons. It is therefore proportional to Z^2 where Z, the atomic number, is also the number of protons. Note that this energy is inversely proportional to the radius and therefore to $A^{1/3}$ and is positive. The fourth term represents the effect of the Pauli exclusion principle. In quantum theory this principle excludes two identical fermions from being in the same state. Or, equivalently they cannot be at the same point. For this reason even bringing two neutrons near each other costs energy. So it would be energetically expensive to pack only neutrons to form nuclei, or only protons. The ideal mixture from the point of view of the exclusion energy would be to have an equal number of neutrons and protons. An unequal mixture would cost more energy. The last term, proportional to D^2 represents this. Recall that $D = A - 2Z$ is the neutron-proton number difference.

We have gotten very far using very simple arguments. This is just by good fortune. Some features of the nuclear system, such as hard core in the N-N force as well as its short range have made such a semi-classical picture possible. But we cannot go much further without much more sophisticated analysis. A theoretical understanding of detailed properties of nuclei, such as their precise energy levels, their wave functions which contain information about their density and charge distributions, their magnetic and other moments, the cross-sections for various nuclear reactions, is beyond the reach of such simple arguments. These properties are important, not just for their scientific value but also for executing the various applications of nuclear science.

In principle, all this detailed information about any nucleus is obtained by solving the Schödinger equation for it's A nucleons.

$$\left(\sum_{i}^{A} -\frac{\hbar^2}{2m_i}\nabla_i^2 + V(1,2, \ldots A)\right)\Psi(1,2, \ldots A) = E\Psi(1,2, \ldots A)$$

where $\Psi(1,2, \ldots A)$ is the wave function in terms of the A coordinates, E the total energy and $V(1,2, \ldots A)$ is the interaction given by

$$V(1,2, \ldots) = \sum_{i<j} v_{ij} + \sum_{i<j<k} w_{ijk} + \ldots$$

where v_{ij} is the normal two-body interaction between the i th and the j th nucleon (including both the strong nuclear as well as the Coulomb interaction), w_{ijk} is the intrinsic 3-body interaction mentioned earlier, and the dots represent the sum of similar 4-body and higher forces.

Now, although basic meson field theory leads us to expect strong 3-body and higher many body forces, reliable expressions for these forces are not available either theoretically or experimentally. Therefore as a starting point let us assume that only two-body interactions v_{ij} are present. Solving the n-body Schrödinger differential equation in terms of just the two-body interactions is difficult enough for large n.

Nuclear Matter (at last!)

Although n-body dynamics is exceedingly difficult for any finite $n > 2$, it becomes in some ways simpler for $n \to \infty$ if we can exploit the translation invariance of that system. Therefore we hypothesize something called Nuclear Matter, which consists of an infinite number of nucleons, distributed at some uniform density over infinite volume. The name is given because it is rather like ordinary matter, like a big chunk of iron made of a large number of identical objects (iron atoms in that case).

Neutron stars are in all likelihood candidates for a physical realisation of nuclear matter.

The nuclear matter system was often called 'hypothetical' because any real nucleus has far from infinite nucleons or infinite radius. So, even if this system is theoretically more tractable than real finite nuclei, how does one compare the results of theoretical properties with experimental numbers? Where can we find a sample of infinite nuclear matter to obtain experimental values for its energy and other properties?

One possibility is to look at the so-called Neutron Stars, which are in all likelihood candidates for a physical realization of nuclear matter. But very little accurate information was available then (or for that matter, now) about the microscopic properties at the individual nucleon level from the bulk properties of neutron stars. We will return to neutron star calculations at the end. But, for an 'experimental test' of any nuclear matter theory we would rather compare its results with numbers inferred by extrapolating from properties of large real nuclei.

To do this, it is easier if we consider nuclear matter that is not only infinite, but has an equal number of protons and neutrons, with the Coulomb interaction neglected. This is called 'symmetric nuclear matter'. In the absence of the Coulomb force, in the understanding

Can one explain the binding energy curve using the Schrödinger equation from first principles?

of whose effects we have some confidence from atomic physics anyhow, the system becomes somewhat more tractable. More importantly an 'experimental' value is available for its binding energy. This is where the Bethe–Weiszacker formula comes in. In the formula, the second term vanishes as compared to the first as $A \to \infty$. The third term is zero since the Coulomb interaction is, by hypothesis absent. Finally since symmetric nuclear matter has equal number of protons and neutrons, the last term is also zero ($D = A - 2Z = 0$). So the entire binding energy comes from the first term (the volume term) alone and we know from fitting the binding energy curve that it has a value of 15.78 MeV per nucleon. Another feature of nuclear matter is also available from finite nuclei. The density profiles of nuclei are measured by scattering electrons off them. From the interior density of these nuclei, measured for different values of A, one can deduce what would be the equilibrium density of infinite nuclear matter, after correcting for the absence of Coulomb interaction. The value is 0.17 nucleons per Fermi^3.

This then would be the first goal of nuclear matter theory, to solve the Schrödinger equation from first principles to obtain the binding energy per nucleon for different values of the density and show that the energy as a function of density minimizes at $\rho = 0.17$ F^{-3} at a value of $E/A = -15.78$ MeV. [These numerical experimental values are being constantly improved and updated, but for this pedagogical presentation typical values given here will do.]

It must be emphasized that the goal of constructing the complicated theory of nuclear matter is *not merely* to calculate these two experimentally acceptable numbers. *There are deeper motives.* One wants to perform calculations *ab initio* starting from the primary Schrödinger equation of A nucleons interacting through their basic forces, rather than starting from some intermediate level 'phenomenological' model. Such models, like the famous Shell model, have been very successful in explaining many properties of nuclei. But that success brings with it more questions. The reasons for the success of the Shell model are not easy to understand. Recall that in atoms their shell structure arises from the fact that there is a dominant central force on the electrons due to the heavy and highly charged nucleus at the center. There is no comparable source of central force within the nucleus. As we discussed earlier, all nucleons in the nucleus are on a similar footing. They all interact with one another equally strongly and there is no natural force center inside a nucleus. One of the motivations for an *ab initio* nuclear theory would be to eventually try and explain why the phenomenological models work.

The shell model has been very successful in explaining many properties of nuclei.

There also remains the question of whether intrinsic 3-body forces and higher forces, novel possibilities not present in atomic physics, play a significant role in large nuclei or not. Since there seem to be no clear cut direct qualitative signatures of the 3-body force, its presence can be inferred only through its quantitative contributions to the energy and wave function of large nuclei, or infinite nuclear matter. This again requires setting up a quantitatively reliable and convergent first principles theory of nuclear matter.

We have explained what nuclear matter is, and why it is important to formulate a systematic theory of it. Along the way we have understood many basic features of nuclear physics. The next logical step in our story, its central part, would be to actually describe the theory that was constructed. Unfortunately, the theoretical structure that was developed is far too complex and structurally long to describe here. It would easily fill up a book. It is also not amenable to being summarized easily. We will therefore be content to mention, in descriptive style, some of the intrinsic difficulties the problem poses and the very clever methods that were devised to

Apart from Bethe, numerous people worked on the nuclear matter problem. Notable amongst them were Keith Brueckner and Jeffrey Goldstone.

overcome those difficulties. Even this will require going to a somewhat more advanced level of presentation than what we have gotten away so far, and fluency with basic quantum mechanics will be unavoidable beyond this stage.

Brueckner–Goldstone Formalism

Apart from Bethe, numerous people worked on the nuclear matter problem. Notable amongst them were Keith Brueckner and Jeffrey Goldstone. Brueckner pioneered the study of nuclear matter theory [1]. Recall that for usual finite bound systems in quantum mechanics, one cannot perturb with respect to the entire potential, i.e. by starting with just the kinetic energy as the unperturbed Hamiltonian H_0. The bound state is generally non-perturbative. This is because the exact eigenstates of the bound system are localized, falling exponentially to zero at large distances, whereas the eigenstates of kinetic energy are plane waves which are non-vanishing throughout space. Thus in most of space, the exact and the unperturbed wave functions would be quite different, violating the spirit of perturbation. But, in the case of infinite nuclear matter, even the bound system stretches all the way to infinity. Hence perturbation theory starting with planes wave is permissible. *This is a very major advantage of infinite matter and translation invariance.* So the unperturbed ground state is just the filled Fermi sea of plane waves up to a Fermi momentum k_F, which is related to the density by $\rho = (2/3\pi^2)k_F^3$. For the experimental value of $\rho = 0.17$ F^{-3}, the corresponding Fermi momentum is $k_F = 1.36$ F^{-1}. This unperturbed ground state is unique i.e. non-degenerate. Starting with this one does standard non-degenerate perturbation theory for the ground state, where all the intermediate states are again n-particle plane waves.

This is 'standard' but one must remember that we have n-particle-states with $n \to \infty$, whose wave function has to be duly anti-symmetrised as a Slater determinant. We also have a Hamiltonian which has an infinite number of pair-wise potentials. A systematic enumeration of all the terms in this perturbation series, with a convenient diagrammatic representation, was derived by Goldstone, in a paper that is a milestone in fermion many-body theory [2].

The nucleon–nucleon potential has a strong attractive part and an even stronger repulsive core.

Like any other perturbation expansion, the Goldstone expansion is an infinite series in powers of the interaction potential v. If the potential had been weak, one could approximate by calculating only the first few terms. But we have seen that the nucleon-nucleon potential is far from weak. It is has a very strong attractive part and an even stronger repulsive core, often represented by an infinitely strong hard core. Not only will the perturbation series not converge in powers of v, but even individual terms will be very large, if not infinite in the hard core limit.

Brueckner overcame this problem by defining his 'Reaction Matrix'. The principle behind this is that even when a potential has infinitely large repulsive portions, it does not mean that the physical quantities will have any actual divergences. The wave function of the system will avoid the regions where the potential is very repulsive. For example consider the simple one particle Schrödinger equation,

$$(H_0 + v)\psi = E\psi$$

In this, even though the potential v may be infinite in some region, the wave function does not diverge. One merely sets it equal to zero where $v = \infty$. (Recall the simple problem of a particle in a box in quantum mechanics textbooks where the potential is taken to be $+\infty$ at the walls. All that the infinite potential outside the box does is to make the wave function vanish at the walls of the box. But there are no infinite potentials in the region where the wave function is *non-vanishing* and so, Bethe and Goldstone used very ingenious approximations to evaluate the G matrix by solving a Schrödinger-like differential equation which came to be known as the Bethe–Goldstone equation.

no infinities in the energy levels.)

In such a situation, it is clearly better to directly solve for the wave function in coordinate space rather than calculate it term by term in perturbation expansion.

Conversely, if one started with a perturbation series involving a strong potential it would be better to try to evaluate its sum in some closed form. Accordingly, Brueckener attempted to numerically sum in closed form ladders of repeated v-interactions between a pair of particles and called it the Reaction matrix G. Goldstone showed that his exact expansion in powers of the primary interaction v can be recast as a set of diagrams involving only the reaction matrix G. Bethe and Goldstone used very ingenious approximations to evaluate the G matrix by solving a Schrödinger-like differential equation which came to be known as the Bethe–Goldstone equation [3].

The work of Brueckner, Goldstone and Bethe thus tamed the repulsive hard core in the original N-N potential. It replaced the perturbation expansion in powers of the original potential by an exactly equivalent diagrammatic expansion in terms of the G matrix. Each term in the expansion was finite and people began to calculate the first few diagrams in the expansion hoping the series would converge fast.

However this turned out to be far from the end of the theory, but merely its first phase. It brings us roughly half way to the ultimate form of the so-called Brueckner–Bethe theory of nuclear matter. The remaining half gets more intricate and we can only mention its highlights here.

Although each term in the Brueckner expansion was finite, the hope that the expansion would converge in powers of the G matrix turned out to be false. This expansion consists of diagrams of ever increasing size and complexity, the typical diagrams involving an arbitrary number of particles interacting repeatedly through pairwise G-interactions. But, using the methods developed by Bethe and co-workers [3,4] to estimate G matrices, it was shown that you could get some hold on the state of convergence of the Brueckner–Goldstone expansion [5]. The key was to look at diagrams of higher and higher order in the G-matrix but which correspond to a fixed number of particles interacting with each other in all possible ways. It was shown that these diagrams remained roughly of the same magnitude. In other words, the Brueckner–Goldstone expansion does not converge for nuclear matter. Therefore the following alternate approach to the theory was suggested [6].

1. Rather than calculate G-matrix diagrams one by one, the sum of all diagrams of all orders in G, but involving only some 3 particles out of the Fermi sea (the 3-body cluster) should be somehow summed in closed form. It was suggested that this be done by converting their sum into a 3-body differential equation rather like the 2-body Bethe–Goldstone equation.

2. Similarly all n-body clusters for any given $n > 3$, but to all orders in G also be summed in closed form. In other words the theory should be cast not as a perturbation theory in powers of the basic interaction v or even the reaction matrix G, but as a cluster expansion,

3. The sum of all n-body cluster diagrams should give a result proportional to $A\rho^{n-1}$. If the density ρ is small enough then the sequence of n-body energies for increasing n may converge. This will happen if $\rho R^3 \ll 1$, where ρ is the particle density and R represents some characteristic size of the inter-particle correlations. We could consider as candidates for R the hard core radius, or the healing distance of the 2-body wave function or the range of the full N-N potential. These typically have values of 0.5 F, 1.0 F, and 1.5 F respectively. With ρ taken

as 0.17 F^{-3}, they give $\rho R^3 \cong 0.02$, 0.17 and 0.57 respectively. These factors, indicating the possible rate of convergence of our re-formulated cluster expansion, are less than unity, but not overwhelmingly so. They can be considered encouraging but only marginally and can be easily offset by details, such as combinatory factors, factors of 2π, etc. Therefore only the actual evaluation of the 3-body, 4-body cluster energies will give us a more reliable estimate of how well the expansion converges.

Bethe then undertook the difficult task of explicitly summing and evaluating the 3-body energy [7]. Using methods devised by Faddeev [8], he set up a procedure for calculating the 3-body energy. Day [9] and Kirson [10] further improved Bethe's method and finally reliable estimates began to be available for the 3-body cluster energy. This process took several years and unfortunately is too complicated to be usefully described here. Interested readers can read a review article summarizing these developments [11]. But the results were very satisfactory.

Day and Wiringa [12] give the results of the 2-.3- and 4-body cluster energies for different values of the density $\rho = (2/3\pi^2)\ k_F^3$. At $k_F = 1.4\ F^{-1}$, very close to the observed k_F of 1.36 F^{-1}, the 3-body energy adds up to $D_3 = (-4.80)$ MeV and the 4-body energy to $D_4 = (-0.64)$ MeV. Comparing these with the 2-body energy of (-35.12) MeV we see a decent rate of convergence. Further the convergence is backed by sound physical arguments and intuition discussed earlier and is not fortuitous. The total of the 2, 3 and 4 body energies is thus (-40.56) MeV. But remember that this is just the interaction (potential) energy of the ground state. To this we have to add the kinetic energy of the filled Fermi sea at that density which is T = 24.39 MeV. Altogether, the total energy per particle comes out as -16.2 MeV, with a calculation uncertainty of about 1 MeV, a range that includes the experimental value from the binding energy curve.

Lastly, we come to the issue of intrinsic three- and many-body forces, mentioned earlier. Fortunately, once the ground state wave function of nuclear matter in the presence of just the familiar 2-body N-N force was understood, arguments could then be given that although three body forces may be large in magnitude, their effect on nuclear matter energy will be largely suppressed [13]. This happens because the wave function has correlation holes in regions of many-body space where three-body forces are large, i.e. when two of them come close together. This is because of the hard core in the N-N interaction. Consequently the effect of three-body forces is expected to be at best of order 1 MeV. Therefore the agreement with experimental values of nuclear matter theory described above will not be seriously ruined by many-body forces.

Given all this, it is fair to conclude that the theory of nuclear matter, starting from first principles, has been satisfactorily constructed after nearly two decades of effort. We conclude by briefly mentioning a few other developments related to our discussion.

Nuclear matter energy has also been calculated using Variational methods. It is different from the Brueckner–Bethe theory. We will not describe it here. But the principles of variational methods are conceptually very straightforward. They are exactly the same as given in quantum mechanics textbooks. You minimize the variational energy

$$E_{Var} \equiv \frac{\langle \Psi_{Var} | H | \Psi_{Var} \rangle}{\langle \Psi_{Var} | | \Psi_{Var} \rangle},$$

where

$$|\Psi_{Var}\rangle$$

is a trial wave function for A nucleons into which you incorporate as much correlations as you can, carrying

Suggested Reading

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parameters. The rest is a matter of just evaluating the integrals in the numerator and the denominator. This is however complicated in the infinite-A limit since both the Hamiltonian and the wave function have infinitely many terms. Therefore techniques from Statistical Mechanics like hyper-netted chains have to be employed to systematically enumerate the terms. The method yields very good numerical results. Interested readers can find more in the papers of Vijay Pandharipande and co-workers [14].

The developments of nuclear matter techniques also led to other sophisticated many-body theory formulations, in particular to the Coupled Cluster Methods used extensively in atomic physics and physical chemistry.

The success of nuclear matter theory has led to applying similar methods to study the interior of pulsars. They are believed to be neutron stars, since they are created as remnants of stars compressed after a supernova explosion into objects just a few kilometers in radius, but with stellar mass. That implies a state of compression to nuclear densities, making pulsar interiors possible real-life candidates for (almost) infinite nuclear matter. One major handicap with applying these ideas to neutron stars is that while a lot of information is available about their bulk stellar properties, little is known directly from observation about their microscopic structure nucleon level. But at the theoretical level a great deal of work has been done on extending nuclear matter calculations to neutron stars. Nuclear matter methods have also been applied to ensembles involving other baryons and also to quark-matter.

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