Classical Sets and Non-Classical Sets: An Overview

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Our traditional models for formal modeling, reasoning or computing are deterministic and precise in character. But real-life situations that we come across are generally non-deterministic and cannot be described precisely. In 1923, the philosopher Bertrand Russell referred to this situation when he wrote:

> *All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life but only to an imagined celestial existence.*

Mathematicians, logicians, and computer scientists are trying to model uncertain, imprecise or vague concepts. Here we present two models of vague concepts and draw a comparison between such imprecise sets and the standard classical sets. In Section 1, we define classical sets, which model precise concepts. The chief features of such sets are also stated. Then in Sections 2 and 3, we define the notions of *fuzzy set* and *rough set*, which model imprecise statements.

1. Crisp Sets

We begin with the notion of a *classical* or *crisp* set. A crisp set is defined in such a way that all the individuals in a given universe can be partitioned into two classes: those who belong to the set, and those who do not belong to the set. Mathematically, we give the following definition.

Keywords
Fuzzy sets, crisp sets, rough sets, law of excluded middle, DeMorgan's laws.

> *If $\mathcal{U}$ is the universe, then the set of elements in $\mathcal{U}$ having property $\mathcal{P}$ (the property is such*

that each element of the universe either has the property or does not have the property) is denoted by B, and can be written as

$$B = \{x \ : \ x \in \mathcal{U} \text{ and } x \text{ has property } \mathcal{P}\} \qquad (1)$$

If A and B are two sets of the universe $\mathcal{U}$, then A is said to be a *subset of* (or contained in) B, denoted by $A \subset B$ or $B \supset A$, if and only if $x \in A \Longrightarrow x \in B$. Two sets A and B of the same universe $\mathcal{U}$ are said to be *equal* if $A \subset B$ and $B \subset A$.

A set consisting of no elements is said to be a *null set*; it is denoted by Φ. Note that a null set is a subset of every set, and any set is contained in its universe.

In order to generate new sets, we have the following definitions.

- The *union* of two sets A and B, denoted by $A \cup B$, is the set of all objects which are members of either A or B, that is,

$$A \cup B = \{x \ : \ x \in A \text{ or } x \in B\} \qquad (2)$$

- The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of all objects which are members of both A and B, that is,

$$A \cap B = \{x \ : \ x \in A \text{ and } x \in B\} \qquad (3)$$

- The *complement* of a set A, denoted by A', is the set of all elements which belong to the universe $\mathcal{U}$ but not to A itself, that is

$$A' = \{x \ : \ x \in \mathcal{U} \text{ and } x \notin A\} \qquad (4)$$

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Bertrand Russell

Zadeh in 1965 proposed a completely new approach to vagueness called *fuzzy set theory*. In his approach, an element can belong to a set to a degree k ($0 \le k \le 1$), in contrast to classical set theory where an element must definitely belong or not to a set.

2. Properties of Crisp Set Operations

Involution	$(A')' = A$
Commutativity	$A \cup B = B \cup A$,
	$A \cap B = B \cap A$
Idempotence	$A \cup A = A$,
	$A \cap A = A$
Identity	$A \cup \Phi = A$,
	$A \cap \mathcal{U} = A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$,
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
Law of Contradiction	$A \cap A' = \Phi$
Law of Excluded Middle	$A \cup A' = \mathcal{U}$
De Morgan's Laws	$(A \cup B)' = A' \cap B'$,
	$(A \cap B)' = A' \cup B'$

3. Fuzzy Sets

In real life we come across many situations where inclusion and non-inclusion in a set are not clearly defined; for example, the classes of *tall people*, *beautiful paintings*, *expensive cars*, *sunny days*, etc. The boundaries of such sets are vague, and the transition from member to non-member appears gradual rather than abrupt. Lofti Zadeh in 1965 proposed a completely new approach to vagueness called *fuzzy set theory*. In his approach, an element can belong to a set to a degree k ($0 \leq k \leq 1$), in contrast to classical set theory where an element must definitely belong or not to a set. The mathematical definition is given below:

If X is a collection of objects, then a fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x)) \ : \ x \in X, \ \mu_A(x) : X \to [0,1]\} \quad (5)$$

where $\mu_A(.)$ is called the *membership function of* A, and is defined as a function from X into $[0,1]$. The membership function gives a 'grade of membership' of the element to the set.

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Note that choice of a different membership function may change the set. If $\mu_A(x)$ can take only the values 0 and 1, then the set is a crisp set.

The analogues of the set-theoretic notions of equality, subset, null set, etc., are defined as follows.

- If the membership grade of each element of a fuzzy set A in X is less than or equal to its membership grade of the same element of a fuzzy set B in the same set X, then A is said to be a *subset* of B. Thus, if for every $x \in X$, we have

$$\mu_A(x) \leq \mu_B(x), \tag{6}$$

then we write $A \subset B$.

- If the membership grade of each element of a fuzzy set A in X is equal to its membership grade of the same element of a fuzzy set B in the same set X, then A is said to be *equal* to B. Thus, if for every $x \in X$, we have

$$\mu_A(x) = \mu_B(x), \tag{7}$$

then we write $A = B$.

- A null set is denoted by Φ, and is that fuzzy set for which the membership grade for each element is zero. Thus,

$$\Phi = \{(x, \mu_\Phi(x)) \ : \ x \in X, \ \mu_\Phi(x) = 0\} \tag{8}$$

- Similarly, the universal set $\mathcal{U}$ is

$$\mathcal{U} = \{(x, \mu_\mathcal{U}(x)) \ : \ x \in X, \ \mu_\mathcal{U}(x) = 1\} \tag{9}$$

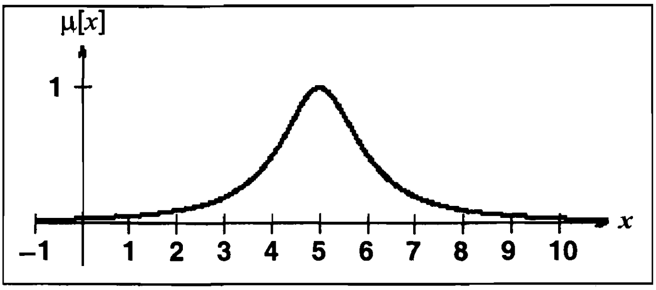

Figure 1. Graph of the membership function $\mu(x) = 1/(1 + (x-5)^2)$.

Example 1. Let X be the set of all real numbers, and let A be the set of all real numbers close to 5. It is apparent that the set is not a crisp set. It can, however be defined as a fuzzy set as follows: $A = \{(x, \mu_A(x)) : x \in X\}$, where

$$\mu_A(x) = \frac{1}{1 + (x-5)^2}.$$

Since $0 \leq \mu_A(x) \leq 1$ for all real numbers x, it is a possible membership function. The graph of the membership function is shown in *Figure* 1.

Example 2. Let X be the set of all books, and let B be the set of thick books. As earlier, this is not a crisp set. We may define it as a fuzzy set by giving the following membership grade function: if x is the number of pages in the book, then

$$\mu_B(x) = \begin{cases} 0 & \text{if } x < 300, \\ (x-300)/200 & \text{if } 300 \leq x \leq 500, \\ 1 & \text{if } x > 500. \end{cases}$$

The graph of the membership function is shown in *Figure* 2.

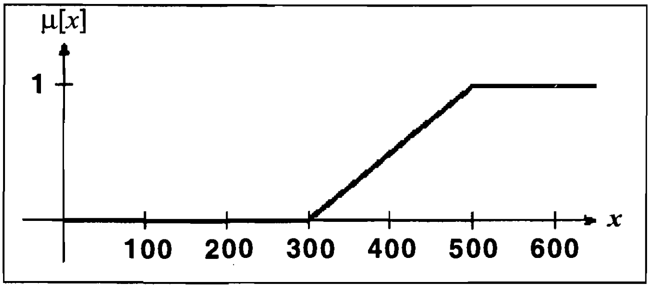

Figure 2. Graph of membership grade function of set of thick books, with x = no. of pages.

3.1 *Operations Defined on Fuzzy Sets*

In order to generate new fuzzy sets, Zadeh gave the following definitions.

- If X is a collection of objects, and A is a fuzzy set in X, then A', the *complement* of A, is a set of ordered pairs, defined by:

$$A' = \{(x, 1 - \mu_A(x)) \ : \ x \in X\} \qquad (10)$$

The union of two fuzzy sets A and B in X is defined by the function $\mu_{A \cup B}(.)$ where

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \qquad (11)$$

and the *intersection* of two fuzzy sets A and B is defined by the function $\mu_{A\cap B}(.)$, where

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\} \qquad (12)$$

4. Properties of Fuzzy Set Operations

Involution
$$(\mu'_A(x))' = 1 - \mu'_A(x) = \mu_A(x)$$

Commutativity
$$\mu_A(x) \cup \mu_B(x) = \max\{\mu_A(x), \mu_B(x)\} = \mu_B(x) \cup \mu_A(x),$$
$$\mu_A(x) \cap \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\} = \mu_B(x) \cap \mu_A(x)$$

Idempotence
$$\mu_A(x) \cup \mu_A(x) = \max\{\mu_A(x), \mu_A(x)\} = \mu_A(x),$$
$$\mu_A(x) \cap \mu_A(x) = \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$$

Identity
$$\mu_A(x) \cup \mu_\Phi(x) = \max\{\mu_A(x), \mu_\Phi(x)\} = \mu_A(x),$$
$$\mu_A(x) \cap \mu_{\mathcal{U}}(x) = \min\{\mu_A(x), \mu_{\mathcal{U}}(x)\} = \mu_A(x)$$

The law of contradiction that holds true for crisp sets is no longer valid for fuzzy sets.

The property of *Associativity* holds, because

$$\begin{aligned}(\mu_A(x) \cup \mu_B(x)) \cup \mu_C(x) &= \max\{\mu_A(x), \mu_B(x), \mu_C(x)\} \\ &= \mu_A(x) \cup (\mu_B(x) \cup \mu_C(x)),\end{aligned}$$

$$\begin{aligned}(\mu_A(x) \cap \mu_B(x)) \cap \mu_C(x) &= \min\{\mu_A(x), \mu_B(x), \mu_C(x)\} \\ &= \mu_A(x) \cap (\mu_B(x) \cap \mu_C(x)),\end{aligned}$$

and the property of *Distributivity* holds too, because $\mu_A(x) \cap (\mu_B(x) \cup \mu_C(x))$

$$\begin{aligned}&= \min\{\mu_A(x), \max\{\mu_B(x), \mu_C(x)\}\} \\ &= \max\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(x), \mu_C(x)\}\} \\ &= (\mu_A(x) \cap \mu_B(x)) \cup (\mu_A(x) \cap \mu_C(x))\end{aligned}$$

Importantly, however, the law of contradiction that holds true for crisp sets is no longer valid for fuzzy sets, as

$$\mu_A(x) \cap \mu'_A(x) = \min\{\mu_A(x), \mu'_A(x) =$$

$$\min\{\mu_A(x), 1 - \mu_A(x)\}$$

is not necessarily equal to 0. Similarly, the law of the excluded middle and De Morgan's laws are not valid. So with the generalization of classical sets to fuzzy sets, we lose some important properties.

5. Rough Sets

Rough set theory is yet another approach to vagueness; it was introduced by Zdzislaw Pawlak in 1982. Here, a vague concept is replaced by a pair of precise concepts called the *lower* and *upper approximations* (analogous to giving lower and upper bounds for a real number whose value is not known with certainty). The lower approximation of a concept consists of all objects which *definitely* or *necessarily* belong to the concept, while the upper approximation of the concept consists of all objects which *possibly* belong to the concept. The difference between the two approximations is a boundary region concept consisting of all those objects whose classification in relation to the concept or its complement cannot be done with certainty, using only available knowledge. The greater the boundary region, the vaguer the concept. If the boundary region is empty, then the concept is precise.

The lower approximation of a concept consists of all objects which *definitely* or *necessarily* belong to the concept, while the upper approximation of the concept consists of all objects which *possibly* belong to the concept.

Suppose we have some information about the elements of $\mathcal{U}$, the universe of discourse. Elements with the same features (as per the given information) are indiscernible and form blocks that may be thought of as 'elementary granules of knowledge' about the universe. Let us assume that the indiscerniblity relation is represented by an equivalence relation $R \subset \mathcal{U} \times \mathcal{U}$. The elements within each block or equivalence class of R are indistinguishable, and are said to form a *granule*. These granules may or may not specify a given subset X of the universe $\mathcal{U}$ exactly. So the set is specified, instead, by two crisp sets which bound it and can be exactly specified by the granules. (See the sketch in *Figure* 3; the small squares are the granules of knowledge.)

- The *R-lower approximation of* X is denoted by $R_{\text{low}}(X)$, and is the set of all those elements which definitely belong to X. So,

$$R_{\text{low}}(X) = \bigcup_{x \in \mathcal{U}} \{R(x) \ : \ R(x) \subset X\} \qquad (13)$$

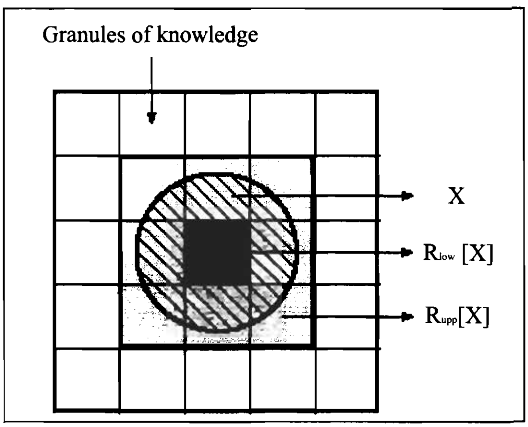

Figure 3. Rough sets.

- The *R-upper approximation of X* is denoted by $R_{\text{upp}}(X)$, and is the set of all those elements which possibly belong to X. So,

$$R_{\text{upp}}(X) = \bigcup_{x \in \mathcal{U}} \{R(x) \ : \ R(x) \cap X \neq \Phi\} \quad (14)$$

- The *R-boundary region of X* is denoted by $BN_R(X)$ and is given by

$$BN_R(X) = R_{\text{upp}}(X) \setminus R_{\text{low}}(X). \quad (15)$$

  Note that X is a crisp set if and only if $BN_R(X)$ is an empty set.

- A rough set X is said to be contained a rough set Y in $\mathcal{U}$ if and only if

$$R_{\text{low}}(X) \subset R_{\text{low}}(Y) \quad \text{and} \quad R_{\text{upp}}(X) \subset R_{\text{upp}}(Y). \quad (16)$$

- Two rough sets X and Y are said to be equal if and only if

$$R_{\text{low}}(X) = R_{\text{low}}(Y) \quad \text{and} \quad R_{\text{upp}}(X) = R_{\text{upp}}(Y). \quad (17)$$

Thus, a rough set is defined as a pair of crisp sets, $(R_{\text{low}}(X), R_{\text{upp}}(X))$.

5.1 *Properties of Rough Sets*

1. $R_{\text{low}}(X) \subset X \subset R_{\text{upp}}(X)$
2. $R_{\text{low}}(\Phi) = \Phi = R_{\text{upp}}(\Phi)$
3. $R_{\text{low}}(\mathcal{U}) = \mathcal{U} = R_{\text{upp}}(\mathcal{U})$
4. If $X \subset Y$, then $R_{\text{low}}(X) \subset R_{\text{low}}(Y)$ and $R_{\text{upp}}(X) \subset R_{\text{upp}}(Y)$.
5. $R_{\text{low}}(R_{\text{low}}(X)) = R_{\text{upp}}(R_{\text{low}}(X)) = R_{\text{low}}$

6. $R_{\text{upp}}(R_{\text{upp}}(X)) = R_{\text{low}}(R_{\text{upp}}(X)) = R_{\text{upp}}$
7. $R_{\text{low}}(X') = (R_{\text{upp}}(X))'$
8. $R_{\text{upp}}(X') = (R_{\text{low}}(X))'$
9. $R_{\text{upp}}(X \cup Y) = R_{\text{upp}}(X) \cup R_{\text{upp}}(Y)$
10. $R_{\text{low}}(X \cup Y) \supset R_{\text{low}}(X) \cup R_{\text{low}}(Y)$
11. $R_{\text{upp}}(X \cap Y) \subset R_{\text{upp}}(X) \cap R_{\text{upp}}(Y)$
12. $R_{\text{low}}(X \cap Y) = R_{\text{low}}(X) \cap R_{\text{low}}(Y)$

A similarity to the axioms and properties of a topological space may be seen in this listing. A 'rough membership function' may be defined on $\mathcal{U}$, namely:

$$\mu_X^R(x) = \frac{|X \cap R(x)|}{|R(x)|}. \tag{18}$$

6. Applications

Any activity in which *judgement* has to be used to make decisions implicitly uses fuzzy logic, but it is not clear how we might make machines do this for us. But this breakthrough has been accomplished, making use of the versatility of electronic circuitry, and commercial applications that use fuzzy logic are now starting to become available in the market. Examples of such applications are devices that can recognize handwriting and speech, and robots that can drive cars – though, as yet, not on Indian roads!

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Smart Washing Machines: Some models of washing machines use fuzzy logic to respond to initial conditions (how dirty the clothes are, what kind of water is used, and so on – these being assessed using sensor technology) and decide on the optimal water quantity and temperature, spin speed, foam control, rinse time, and so on. Some models even weigh the load, advise how much detergent to put in, assess material type and water hardness, and check if the detergent is a powder or a liquid. (A few still more advanced models are available that apparently even learn from past experience. If the advertisements are to be believed, soon we should be able to send these washing machines to university to make further advances in fuzzy set theory!)

Rough Sets: The algebraic and topological foundations of rough sets have attracted a fair amount of attention in recent years. Connections with probability theory, fuzzy set theory, the theory of evidence, boolean reasoning and decision theory, among others, have been explored. Applications have been pursued in fields as varied as pharmacology, medicine, banking, market research, engineering and speech recognition. The promise of applications to fields such as information retrieval, neural networks, machine learning and data mining seems quite real at the present moment.

7. Conclusion

We find from the above discussion that as we try to introduce imprecision, we lose precise results. The union and intersection operators in classical set theory give rise to a well formed structure. With the introduction of vagueness, we have to compromise with a few results, as is only to be expected.

As computer scientists are trying to model human brain in AI (Artificial Intelligence), the significance of such non-classical sets is increasing day by day. We look forward to the day when a computer will be able to read the mood of its user and act accordingly.

Suggested Reading

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