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Two Dimensional Potential Mapping – Monte Carlo Simulation

A very useful experiment of two dimensional potential mapping, namely electrolytic tank model, for graduate and post graduate level physics students is given here. Laplace's equation is solved for the above and the results are compared with the experiment. The agreement is so good that this is extended to complex problems. Monte Carlo simulation, an alternative to the above experiment, is developed for these complex problems and compared with the experimental results of electrolytic tank model.

1. Introduction

Potential at any point in a system, whether it is a charged sphere or hollow sphere or cylinder etc., both inside or outside the boundary, is obtained by solving Laplace's equation $\nabla^2\phi = 0$ with suitable boundary conditions [1]. If this could be demonstrated experimentally in the laboratory, then students would derive more insight into the problem. For this, a simple experimental set-up is explained and the measurement is compared with the theoretical (Laplacian) value. A simple two dimensional configuration is considered for the experiment.

Computer simulation replaces every laboratory experiment in one way or another, as an in-depth study of a problem is normally not possible in the conventional laboratory experiment. So, an attempt is made here to verify the experimental result and Laplace's result by simulation, considering only simple two dimensional problems. Then complex configurations will be considered for experiment and simulation to demonstrate the usefulness of simulation.

Keywords
Potential map, Laplace's equation, Monte Carlo simulation.

This article can be broadly divided into two independent

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problems.

- To find a suitable experimental substitute for Laplace's equation.
- To extend this for complex problems.

2. Laplace Equation

Whenever the electric potential V corresponding to a given charge distribution is required, the solution to Laplace's or Poisson's equation satisfying the required boundary condition will be found, i.e. the potential ϕ at any point in any configuration is calculated from the Laplace's equation $\nabla^2\phi = 0$ with suitable boundary conditions, for that particular configuration.

Consider an example of two parallel plates separated by a distance D along the x-axis as shown in *Figure* 1. The one-dimensional Laplace's equation can be written as

$$\frac{\partial^2\phi}{\partial x^2} = 0. \tag{1}$$

This may be solved for ϕ as

$$\phi = cx + b, \tag{2}$$

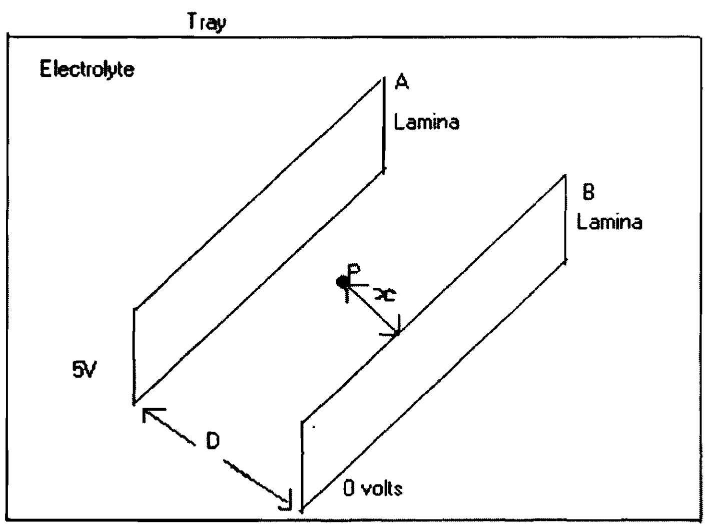

Figure 1. An electrolytic tank model with two brass laminas A and B separated by a distance D.

where c and b are constants, and x is the distance from the reference plate (of the two plates) to the point of interest where potential is required. Now the boundary conditions for this problem of two laminas separated by a distance D are

$\phi = 0$ for $x = 0$ (considered as reference plate, plate B)

$\phi = V$ for $x = D$ (plate A).

These are applied to (1) to get

$$c = \frac{V}{D} \quad \text{and} \quad b = 0.$$

Therefore

$$\phi = \frac{Vx}{D} \tag{3}$$

will be the general solution, when Laplace's equation is solved for this configuration of two parallel plates. This is the analytical result or the expected result from theory.

An attempt is made here to verify the experimental result and Laplace's result by simulation, considering only simple two dimensional problems.

3. Electrolytic Tank Model

What is the experiment to verify this analytical result (3), derived in Section 2? Electrolytic tank model is considered to be an effective experiment to study the potential distribution in any two dimensional system. This is demonstrated here.

Consider the same example of two laminas. The lamina (brass) has a dimension of 10cm × 5cm × 2mm for length, breadth and thickness. Two such laminas are placed in a tray (of non conductor material like plastic) vertically, separated by a distance of D (=10cm). A cm-cm graph sheet is pasted on the bottom of the tray for making measurements on distances. Then, some electrolyte, usually water and common salt, is poured into the tray to a depth of about 1mm (this is the reason why we call it as electrolytic tank model) so that the distribution of the equipotential lines can be measured

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using a multimeter. Two different potentials (+5V and 0V) are applied to the plates A and B respectively. Rigid wooden or plastic (and not metal) clamps should be used to keep the laminas in position. A schematic diagram is shown in *Figure* 1.

Now the potential at any point P between these two plates can be measured by a high impedance voltmeter preferably a digital voltmeter, with respect to the ground (0V), i.e. with respect to the lamina B. This tank model clearly indicates that the measurements are for the x direction only, as these plates are placed parallel to the y direction. Now, the potential at various points inside the tank but between the two laminas are measured in terms of distance x and tabulated (*Table* 1). The error involved in the experimental measurement of voltage at different points is about 2%.

With the Laplacian result (3) given earlier, the potential at the same points where measurements are done, are calculated and given in the same *Table* 1. A good comparison between the theory and the experiment by electrolytic tank model clearly indicates that the experiment can replace the theoretical investigation for finding out the potential at any point in a two dimensional configuration.

Table 1. Two parallel plates separated by a distance. V= 5.1V; D= 10cm (Measured with a digital multimeter). Experimental accuracy is .01 V.

Distance	Potential in volts	
x cm	Experimental	Theory (Laplace) Vx/D
1	0.47	0.51
1.5	0.74	0.76
2	0.97	1.02
4	1.92	2.04
6	2.95	3.06
8	3.96	4.08

In order to justify this claim, we have repeated for other configurations such as two concentric cylinders, two hollow cylinders separated by a distance in the same electrolytic tank model for potential measurement at various points. (These results are not shown here). Every time the comparison with the respective solution of Laplace's equation is found to be very good. Hereafter this experimental setup (electrolytic tank model) will be used for complex two dimensional configurations.

Monte Carlo simulation can be defined as a numerical technique based on a physical or mathematical model where random numbers are used to solve static problems.

4. Monte-Carlo (MC) Simulation

Whenever experiments are complex or experimental facilities are not available in the laboratory, we prefer computer simulation to do the experiment. Monte Carlo simulation can be defined as a numerical technique based on a physical or mathematical model where random numbers are used to solve static problems. That is, time independent activities can be simulated by Monte Carlo method and such simulation is used to mimic the real system, for which the modeling and simulation are done.

A simple example for Monte Carlo is given below [2]. The area of a pond whose shape is irregular can be measured by MC using a pile of stones. Suppose the pond is in the middle of a field of known area A. One way to estimate the area of the pond is to throw the stones so that they land at random within the boundary of the field and count the number of splashes that occur when a stone lands in a pond. The area of the pond F_n is approximately the area of the field times the fraction of stones that makes a splash, ie.,

$$F_n = A\frac{n_s}{n},$$

where n_s is the total number of splashes and n is the total number of throws.

5. Electrolytic Tank and MC Simulation

In this section, we try to study experimentally a com-

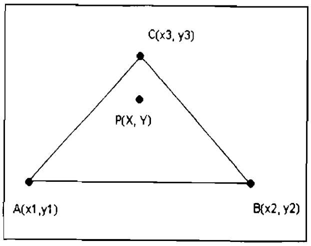


Figure 2. Three laminas forming a closed triangle. P is the point at which potential is evaluated.

plex two-dimensional system for potentials and then do simulation for the same. For example, we want to solve Laplace's equation for the case shown in *Figure* 2, where three laminas are placed in a triangular shape. Finding a theoretical solution for this will be really complex. But measurements are easily possible with the electrolytic tank model as explained earlier, to find the potential distribution inside the three plates.

Three laminas AB, BC, CA of the same dimensions as given earlier are arranged in a triangular shape without touching each other at the corners. These plates AB, BC, CA are given voltages 1.5V, 1.5V, 3V respectively. Electrolyte is taken as usual in the tray. Then voltage at different points (P) within the enclosed triangle (*Figure* 2) is measured and displayed in *Figure* 3. The error involved in the experimental measurement of voltage at different points is around 4%.

If the voltages at various points are required, then we have to solve Laplace's equation in two dimensions

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{4}$$

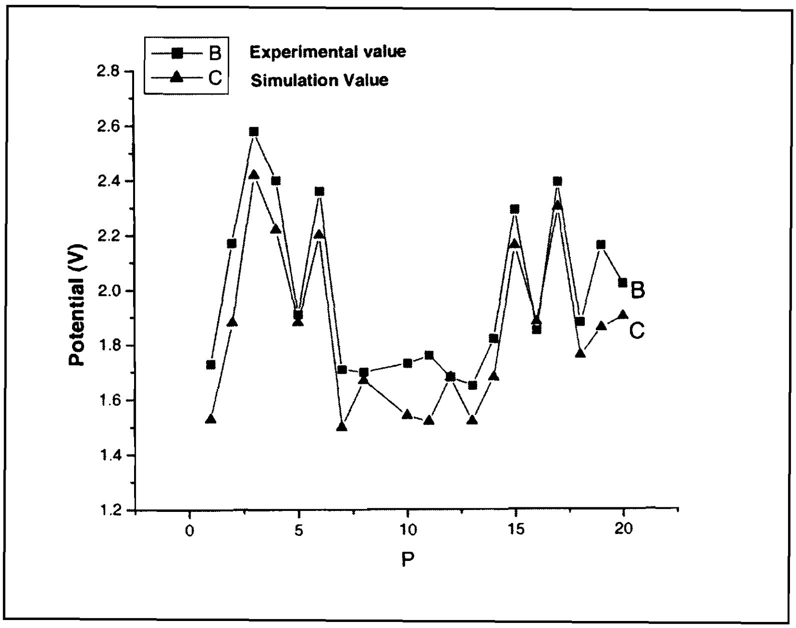


Figure 3. Comparison of experimental and simulation results (triangle).

The boundary conditions for this configuration (*Figure* 4) could not be easily spelt out and this makes the problem difficult. But from the previous example we know very well that the measurements from the electrolytic tank will be alright for the potential distribution as the experimental results reproduce Laplace's results. Similarly we have shown that MC simulation could explain very well the experimental results. So, here we carry out MC simulation to work out the potential distribution rather than solve Laplace's equation.

6. Application of MC Simulation to the Triangular Problem

1. The vertices of the triangle as seen from the electrolytic tank model are taken as the input. Let them be $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$. Actually in the experiment they are (2.5,1), (15,1), (8,12.5). Refer *Figure* 4.

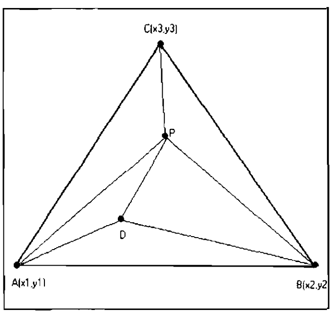
2. Consider a point $P(X, Y)$ where experimental measurement is done and for which simulation is intended.

3. Randomly some point $D(x, y)$ is chosen using the random number generator twice for x and y coordinates.

4. Then it is checked whether it falls inside the triangle ABC or outside it. This is decided as follows. The lengths of the three sides of the triangle are calculated in terms of the coordinates of the vertices and these lengths are denoted by a, b, c. Since the circumference $s = \frac{a+b+c}{2}$, area of triangle $ABC = \sqrt{s(s-a)(s-b)(s-c)}$. If we consider that the random point $D(x, y)$ falls within the triangle ABC, then Area of ΔABC = Area of ΔABD + ΔACD + ΔBCD. If this condition is not satisfied, then $D(x, y)$ lies outside the triangle ABC and that point is rejected.

5. Now, consider the case of point $D(x, y)$ lying within the triangle ABC (termed success). Then we have to find out in which of the three small triangles this point

Figure 4. Three laminas forming a closed triangle. P is the point at which experimental measurement is done. D is the random point.



lies. Let us assume that it lies in triangle ABP as shown in *Figure* 4. If the area of the triangle ΔABP = Area of ΔABD + ΔADP + ΔBPD, then the point D lies inside the ΔABP. If it is not equal, then the same procedure is repeated for ΔBCP and ΔCAP. At least for one of the three cases this condition will be satisfied. In this way, this procedure is repeated for a large number of trials.

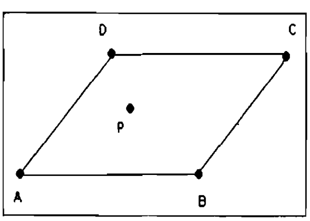
6. Let N_A, N_B, N_C denote the number of successes out of N trials that fall within triangles ABP, BCP, CAP respectively. A suitable program is written for this and executed for different trials from 10,000 to 1,00,000. Then the voltage at point P is calculated, using elementary statistics [3], as

$$V_P = \frac{(N_B N_C V_A + N_A N_C V_B + N_A N_B V_C)}{(N_A N_B + N_B N_C + N_C N_A)}.$$

(This is discussed in detail in Appendix 1). The results from the simulation program along with the experimental results of electrolytic tank are displayed graphically in *Figure* 3. The agreement between the experiment and simulation is quite good, the average deviation being about 7% and the maximum around 14%. This shows that Laplace solution can be very well replaced by simulation and that our modeling and simulation is reliable.

To justify this result another complex problem is considered. Four laminas of the same dimensions as given earlier are arranged in a quadrilateral shape without touching each other at the corners and they are given voltages 1.5V, 3V, 5V,and 6V respectively as shown in *Figure* 5. Electrolyte is taken as usual. Then voltages at different points within the enclosed region are measured and displayed in *Figure* 6. The typical error in these measurements in about 2%. Monte Carlo simulation is done for this quadrilateral in a similar way to that for the triangular configuration.

Figure 5. Four laminas forming a qudrilateral. P is the point at which potential is evaluated.



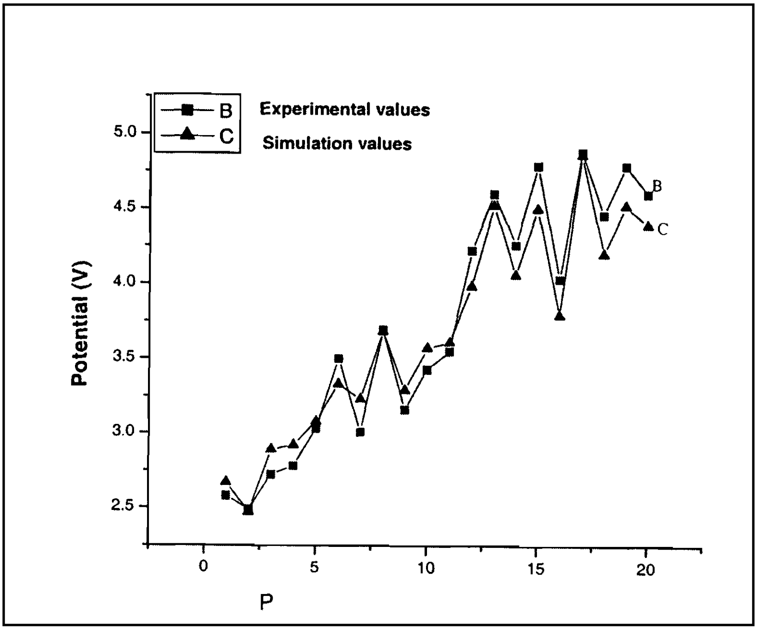


Figure 6. Comparison of experimental and simulation results (quadrilateral).

The potential at point P of the enclosed space can be determined as earlier. Here we will have four triangles (*Figure* 7) within the quadrilateral and so the algorithm given earlier for the triangular case is adapted to this problem. Using elementary statistics [3],

$$V_P = \frac{N_B N_C N_D V_A + N_A N_C N_D V_B + N_A N_B N_D V_C + N_A N_B N_C V_D}{N_B N_C N_D + N_A N_C N_D + N_A N_B N_D + N_A N_B N_C},$$

where N_A, N_B, N_C, N_D denote the number of successes out of N trials that fall within triangles ABP, BCP,

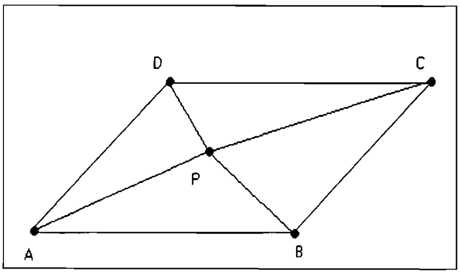


Figure 7. Four laminas forming a quadrilateral. P is the point at which potential is evaluated.

CDP, DAP respectively. Using the above formula, the potential at different points inside the enclosure is determined. These are displayed graphically in *Figure* 6. The error involved in the measurement of voltage by simulation at different points is about 1%. The maximum difference between the experiment and simulation in the quadrilateral configuration is around 7%. So the agreement between them may be considered fairly good.

7. Conclusion

The whole idea of this article is to find a substitute for Laplace's equation $\nabla^2\phi = 0$ in electromagnetic theory.

a) We have proved experimentally that electrolytic tank model can be very well considered as a substitute for $\nabla^2\phi = 0$, i.e., instead of solving the problem theoretically, one can straight away do the experiment to study the potential distribution in a system.

b) We tried this electrolytic tank model experiment for complicated cases, the triangular and quadrilateral configurations, where Laplacian results are not available.

c) Since Monte Carlo simulation is one of the alternative tools to study any complex problem, we proceeded to simulate theoretically the results of triangle and quadrilateral. The model, simulation and the results agree very well with the experimental (electrolytic tank model) measurements.

We conclude by saying that Monte Carlo simulation is the best alternative tool for both theory and experiment. Here we have developed our own software and no readymade software is used. This will help the beginners also to model and simulate simple physical problems. Similarly, the experimental electrolytic tank model can also be practiced for physics students.

Suggested Reading

[1] R Reitz, F J Milford and R W Christy, *Fundamentals of Electromagnetic Theory*, III Edition, Narosa Publishing House, New Delhi, 1998.

[2] Harvey Gould and Jan Tobochnik, *An Introduction to Computer Simulation Methods*, II Edition, Addison-Wesley Publishing Company Inc., 1996.

[3] Donard de Cogan and Anne de Cogan, *Applied Numerical Modelling for Engineers*, Oxford University Press, Inc., New York, 1997.

Appendix 1

The probability of a random event A, is evaluated quantitatively by means of some number

$$n = P(A). \tag{A1}$$

For example, if there are q number of exhaustive, mutually exclusive and equally likely cases of an event and suppose that p of them are favorable to happening of an event A, then the mathematical probability of the event A is defined as

$$P(A) = \frac{p}{q}. \tag{A2}$$

Let us say, in $(m+n)$ number of trials, m trials are in favor of the event A to happen, then

$$P(A) = \frac{m}{m+n}. \tag{A3}$$

Now, we extend this equation (A3) to a case of two infinitely long rods A and B separated by a distance in the x-axis. If N is the total number of times the trial is attempted, then the event (hitting the rod A or the trial point being closer to A than B) happens N_A times and so the probability of event A happen is N_A/N. Similarly on rod B it is N_B/N. Now, let us assume that rod A is applied with a voltage V_A and rod B with a voltage V_B.

At any point on the x-axis between the two rods A and B the voltage will be given in terms of the total probability, as both contributions will determine the voltage at any point x. That is

$$(N_A + N_B)V(x) = N_A V_B + N_B V_A$$

i.e. $N_A V_B$ is the contribution to rod A and $N_B V_A$ to rod B.

Therefore

$$V(x) = \frac{N_A V_B + N_B V_A}{N_A + N_B}. \tag{A4}$$

This may also be proved from the additive law of probability.

Theorem

$$P(A_1 + A_2) = \frac{m_1 + m_2}{N}, \qquad \text{(A5)}$$

where N is the total number of exhaustive, mutually exclusive and equally likely cases of which m_1, m_2 are favourable to the events A_1, A_2, respectively. That is

$$\begin{aligned} P(A_1 + A_2) &= P(A_1) + P(A_2) \\ &= \frac{m_1}{N} + \frac{m_2}{N}. \end{aligned} \qquad \text{(A6)}$$

In the present case of two rods A and B with voltages V_A and V_B the potential at any point x is then

$$P(A + B)V(x) = P(A)V_B + P(B)V_A. \qquad \text{(A7)}$$

Here the events are not mutually exclusive, as the potential on the rods A and B will simultaneously determine the potential at point x.

So from equation (A7)

$$V(x) = P(A)V_B + P(B)V_A$$

as $P(A + B) = 1$.

Similarly for three plates (infinitely long on the z-axis), the potential measured depends on x and y of any point $q(x, y)$

$$V(x, y) = \frac{N_B N_C V_A + N_A N_C V_B + N_A N_B V_C}{N_B N_C + N_A N_C + N_A N_B}.$$