Shailesh A Shirali
Principal
Amber Valley Residential School
K M Road, Mugthihalli
Chikmagalur 577 101, Karnataka
India.
Email:
Shailesh_Shirali@rediffmail.com

Suspension Bridges
An Application of Lamé's Theorem

Introduction

A well-built suspension bridge is a most impressive sight to behold; it is a marvel of both architecture and engineering and has been poetically described as a "symphony in steel." The most famous such bridge is undoubtedly the Golden Gate Bridge in San Francisco, USA. Built in 1937 and with a length of 1280 meters, it is widely regarded as one of the most beautiful bridges ever built (see *Figure* 1). Another, nearly as famous, is the Brooklyn Bridge, in New York City, USA; it was built in 1883 and has a span of 486 meters. The longest such bridge built to date (it was opened only as recently as 1998) is the Akashi Kaikyo Bridge in Japan; it has a length of 1991 meters.

In a suspension bridge the roadway is held by vertical cables or rods that are attached to two curving cables that run along the length of the bridge. These cables pass over a pair of turrets at opposite ends of the bridge and are securely anchored to deeply laid foundations. The basic design of such a bridge has been known for a long time; one of the earliest suspension bridges was built with bamboo in China (where else?) in the 3rd century BC.

Keywords
Suspension bridge, catenary, parabola.

Figure 1. The Golden Gate Bridge, San Francisco.

Courtesy: Mandalagiri S Ravishankar, USA

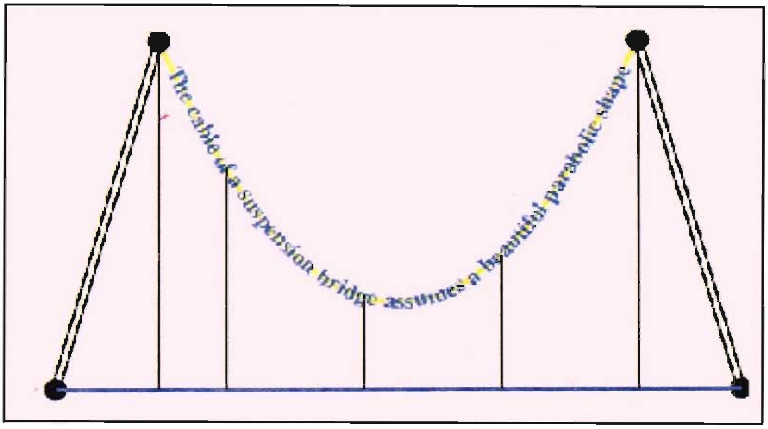

Figure 2. Shape of a suspension bridge.

The vertical rods that hold a suspension bridge obviously need to be extremely strong, if they are to support heavy vehicular traffic. In this article we shall focus on the shape of the suspension cable that holds the bridge via the rods. We shall show that the shape is a *parabola* and not a catenary, as is sometimes supposed. The reason for the mixup should be clear: the shape of a uniform chain freely hanging under its own weight (e.g., a telephone cable) is a catenary (*catena* in Latin means 'chain'), and a catenary closely resembles a parabola in its lower portion. (See *Figures* 2 and 3; the vertical scale has been shown exaggerated.)

The Shape of a Suspension Bridge

We make a few simplifying assumptions about the various cables used in the bridge, and the turrets at the

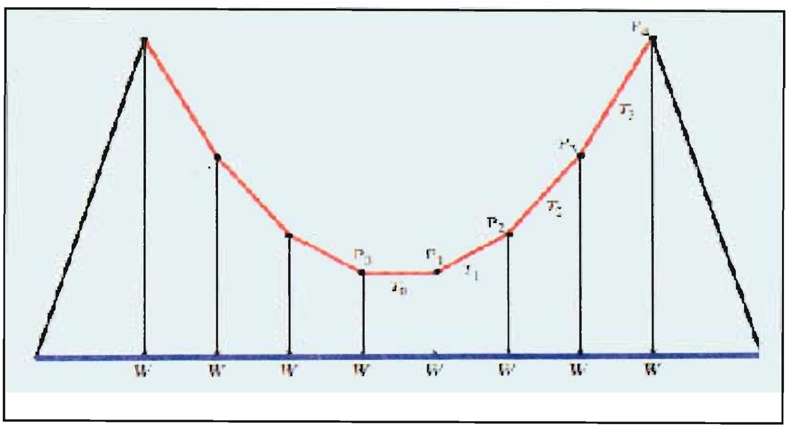

Figure 3. Analysis.

ends of the bridge.

(a) We assume that the vertical rods or cables holding the bridge are placed at equal intervals and bear a uniform load (so the load on the bridge is uniformly distributed across its length).

(b) We take the weights of the cables to be negligible in comparison with the weight of the bridge itself. We also take the cables to be perfectly flexible and inelastic.

(c) We assume that the turrets at the ends of the bridge are of equal height.

Of course, we may take the curve of the suspension cable to be symmetric in the plane that bisects the bridge at right angles, perpendicular to its length.

The well-known (and highly readable) book by Simmons [1] uses these assumptions to formulate a differential equation and then presents a solution. We give a different treatment here, based on one given by Petrov[2]. Let the horizontal distance between any two adjacent rods be $2a$, and let the points where the rods are attached to the cable be P_0, P_1, P_2, In *Figure* 3, the lowermost portion of the cable is shown as segment P_0P_1. Let the height of P_0P_1 above the roadway be b. Let the angle made by P_kP_{k+1} to the horizontal be α_k, for $k = 0, 1, 2,$; of course, $\alpha_0 = 0$. Impose a coordinate system in which the roadway is the x-axis and the vertical line of symmetry of the bridge is the y-axis; then the x-coordinate of P_k is $(2k-1)a$. Let the tension in segment P_kP_{k+1} be T_k, and let the load acting downwards along each vertical cable be W

Lamé's theorem (see [3]) should be well known to students of physics. It states that if three non-collinear forces are in equilibrium, then the ratio of each force to the sine of the angle contained between the other two forces is the same. It is essentially equivalent to the sine rule in plane geometry.'

Using Lamé's theorem[1] on the relationship between three forces in equilibrium, we get:

$$\left.\begin{aligned}\frac{W}{\sin(\alpha_1-\alpha_0)} &= \frac{T_0}{\cos\alpha_1} = \frac{T_1}{\cos\alpha_0},\\ \frac{W}{\sin(\alpha_2-\alpha_1)} &= \frac{T_1}{\cos\alpha_2} = \frac{T_2}{\cos\alpha_1},\\ \frac{W}{\sin(\alpha_3-\alpha_2)} &= \frac{T_2}{\cos\alpha_3} = \frac{T_3}{\cos\alpha_2},\end{aligned}\right\} \quad (1)$$

and so on. From successive pairs of equations, we get:

$$\frac{\sin(\alpha_2-\alpha_1)}{\sin(\alpha_1-\alpha_0)} = \frac{\cos\alpha_2}{\cos\alpha_0}, \qquad \frac{\sin(\alpha_3-\alpha_2)}{\sin(\alpha_3-\alpha_1)} = \frac{\cos\alpha_3}{\cos\alpha_1},$$

and so on, and these relations may be written in more symmetric form as

$$\frac{\sin(\alpha_2-\alpha_1)}{\cos\alpha_2\cos\alpha_1} = \frac{\sin(\alpha_3-\alpha_2)}{\cos\alpha_3\cos\alpha_2} = \frac{\sin(\alpha_4-\alpha_3)}{\cos\alpha_4\cos\alpha_3} = \quad (2)$$

These relations yield: $\tan\alpha_1 = \frac{W}{T_0}$, and

$$\tan\alpha_1 - \tan\alpha_0 = \tan\alpha_2 - \tan\alpha_1 = \tan\alpha_3 - \tan\alpha_2 = \quad \tan\alpha_k = k\tan\alpha_1 = \frac{kW}{T_0} \qquad (k = 1,2,3,\ldots). \quad (3)$$

Let the coordinates of P_k be (x_k, y_k). Using the above, we get:

$$\begin{aligned} y_k &= b + a(\tan\alpha_0 + \tan\alpha_1 + \tan\alpha_2 + \quad + \tan\alpha_{k-1})\\ &= b + a(1+2+3+ \quad + (k-1))\\ \tan\alpha_1 &= b + \frac{a\,k(k-1)}{2}\frac{W}{T_0}. \end{aligned} \quad (4)$$

From $x_k = (2k-1)a$ we get $k = \frac{1}{2}\left(1+\frac{x_k}{a}\right)$, and this yields a quadratic relationship between x_k and y_k:

$$y = \frac{W}{8aT_0}x^2 - \frac{aW}{8T_0} + b. \quad (5)$$

It follows that the points P_0, P_1, P_2, fall on a parabola, as claimed. If the vertical rods are large in number and separated by small intervals, then we may take the shape of the suspension cable to be a smooth parabola.

Suggested Reading

[1] G S Simmons, *Differential Equations*, Tata McGraw Hill.

[2] Y S Petrov, Suspending Belief, in *Quantum*, July-August 1993.

[3] S L Loney, *The Elements of Statics and Dynamics*, Cambridge Univesity Press.

For more on the suspension bridges referred to in the article,here are the URL's of three highly readable websites:

http://www.pbs.org/wgbh/buildingbig/wonder/structure/golden_gate.html
http://www.pbs.org/wgbh/buildingbig/wonder/structure/brooklyn.html
http://www.pbs.org/wgbh/buildingbig/wonder/structure/akashi_kaikyo.html

A human being is part of a whole, called by us the "Universe", a part limited in time and space. He experiences himself, his thoughts and feelings, as something separated from the rest -- a kind of optical delusion of his consciousness. This delusion is a kind of prison for us, restricting us to our personal desires and to affection for a few persons nearest us. Our task must be to free ourselves from this prison by widening our circles of compassion to embrace all living creatures and the whole of nature in its beauty.

– Albert Einstein