Mathematics in Engineering – Part I

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In this two part article I try to convey some of the variety and excitement involved in the application of mathematics to engineering problems; to provide a taste of some actual mathematical calculations that engineers do; and finally, to make clear the distinctions between the applied subject of engineering and its purer parents, which include mathematics and the physical sciences. Two main points of this article are that in engineering it is approximation, and not truth, that reigns; and that an engineer carries a burden of responsibility that mathematicians and scientists are spared. In this first part, I present my views on how engineering differs from mathematics and the physical sciences, and describe what I consider the workhorses of engineering calculations.

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What is Engineering?

Engineers have done a poor job of defining who they are, at least in India.

Most people are aware of the excessively many engineering colleges of our country, producing engineers yearly in lakhs for an economy needing only a fraction thereof.

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However, the employed and capable engineers who design and build rockets, satellites, cameras and missiles for organizations like ISRO or DRDO are called scientists. Engineers who work on industrial R&D for companies like, say, Tata Motors (which recently changed its name from one that included the word 'engineering') are called managers. Engineers who develop brand new products and sell them successfully are called entrepreneurs. Engineers who program computers are called

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Engineering is the name for activity geared towards the purposeful exploitation of the laws, forces and resources of nature for a direct improvement of the human condition.

IT professionals. Any engineer who achieves something risks being called by another name!

Yet, engineering is exciting and worthwhile. Its broad sweep encompasses physics, chemistry, biology, mathematics, economics, psychology and more. It is the name for activity geared towards the purposeful exploitation of the laws, forces and resources of nature, not merely towards uncovering further esoteric truths but towards a direct improvement of the human condition.

In an idle moment, the reader may find it amusing to look on the web for a definition of engineering. I suggest a Google search for 'what is engineering'. Here are three I found:

1. Derived from the Latin ingenium, engineering means something like brilliant idea, flash of genius. The word was created in the 16th century and originally described a profession that we would probably call an artistic inventor.
2. A general term which refers to the systematic analysis and development of measurable physical data, using applied mathematical, scientific, and technical principles, to yield tangible end products which can be made, produced, and constructed.

3. The use of scientific knowledge and trial-and-error to design systems.

Of these the first is, irrationally, gratifying (as if the Latin source makes all engineers ingenious). The second is reasonable but wordy. The third points to the tremendous role of trial and error in design; the multitude of ideas that are tested, with most subsequently rejected; and the systematic use of ideas from proven, successful, designs. I mention this here because the rest of this article is focussed on the 'use of scientific knowledge.

And Science, and Mathematics?

The physical sciences are concerned with the truths of nature, and the laws that govern the world. Mathematics is a beautiful subject that, though inspired by the study of the world, does not depend on it: it can exist by and grow within itself. These subjects are pure. Engineering, in contrast, is *not* pure.

What thickness of reinforced concrete is sufficient for the roof of a 3 metre wide tunnel through a mountain? What is a safe wall thickness for a lead container of given diameter used to carry radioactive waste so that it can, say, sustain a drop from a given height? How many tons of airconditioning are needed to maintain a temperature of 20 degrees Celsius in an office of 200 square meters floor area, 42 people, 35 personal computers, 26 windows and 6 doors, if situated in (say) Hyderabad? How much rocket fuel is needed to transport one kilogram of gold to the moon?

These are *technological* questions, as opposed to scientific or mathematical questions. They are faced by engineers, as opposed to scientists and mathematicians. They seek no fundamental truth about nature, nor require some pure standard of intellectual rigour in their answers. They also supply incomplete information and leave room for the engineer to make simplifying assumptions, develop models, carry out calculations, draw on prior experience, and use safety factors where applicable, to obtain reasonable answers with reasonable effort. And there is a human price to be paid for a wrong answer, be it the loss of life, health, comfort, or gold.

The process of training an engineer to answer such questions requires a study of engineering models and the mathematical techniques used to analyse them. Those models, though approximate, require correspondence with reality in their conception, and precision in their description. And those mathematical techniques, like

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all mathematical techniques, require practice, sophistication and rigour. In this way, the technological world of an engineer builds up from the purer disciplines of mathematics and the sciences, but is not contained in them.

Engineering Mathematics: Everywhere, Everyday

From stress analysis of machine components (using finite element packages), to numerical descriptions of the artist-drawn shapes of new gadgets (using CAD packages), to the use of numbers associated with the mundane jobs of production, inspection, and statistical quality assurance (using statistical packages), to the economically critical planning problem of what material to buy in what amount from where (using optimization packages), and so on, applied mathematics is everywhere in the everyday world of software applications in routine engineering.

From calculations of heat and mass flow in steam power plants and car radiators, to calculations of air flow in cooling fans, to calculations of molten metal flowing and mixing in weld pools, applied mathematics turns the wheels of engineering analysis and design.

From reliability in electrical power system grids to traffic in networks (both tar roads and optical fibres), mathematics crosses boundaries in a way no other technical subject can.

The applications mentioned above are the subjects of many books. Yet, they collectively fail to convey the excitement that engineering applications of mathematics can have. There is more to the story than a list of applications.

The Workhorses of Computational Analyses

Any serious discussion of mathematics in modern engineering must involve the role of computers. Computers are good at moving numbers around and doing arithmetic with them; and they excel at doing these things with large *arrays* of numbers. Coincidentally or consequently, almost all big mathematical problems in engineering are somehow reduced to manipulation of, and arithmetic with, large arrays. In the engineering mathematics context, these arrays are called *matrices.*

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The rest of this section concentrates on matrix calculations. For the nonmathematical reader, let me say this section discusses the three important problems of matrix based calculations. These are called the linear homogeneous system, the standard linear system, and the linear eigenvalue problem. Of these three, the first one is important mostly in understanding the other two problems. The second is central: a large number of applied problems with apparently nothing in common are considered solved when reduced to the standard linear system. The third is independent, and almost as important: it is crucial in understanding vibrations, stability, and more generally sets of solutions that are peculiarly special for the system under study. For a concrete example, consider an engineer designing a bridge: the linear homogeneous system plays a role in deciding whether the bridge can bear loads at all; the standard linear system is used to calculate the deformation or deflection in the bridge when carrying, say, a bus; and the linear eigenvalue problem summarizes vibrations of the bridge.

What follows in this section is somewhat technical. Nontechnical readers may skim or skip as needed.

Three Problems

There is much applied work that can be fruitfully done through an understanding of the below three equations[1]

$$Ax = 0, \tag{1}$$

$$Ax = b, \tag{2}$$

$$Ax = \lambda x. \tag{3}$$

[1] I have left out some other important problems. One certainly is the standard linear program, encountered in optimization. Another, perhaps, is the general nonlinear first order system of ordinary differential equations.

In the above, A is a matrix, x and b are column matrices, and λ is a number (possibly complex). Of these, x and λ are usually unknown, and have to be found when the others are given. No serious computational work in the traditional areas of engineering involving the physical sciences is possible without a good understanding of at least one of these equations. In many advanced problems, one has to use a sequence of steps, each of which has something to do with one of these three equations.

The Linear Homogeneous System (1)

The equation $Ax = 0$, with A an $m \times n$ matrix (m rows, n columns), is usually associated with the question of whether x can be nontrivial (nonzero). If $n \leq m$, then the existence of nonzero x implies that the columns of A are not linearly independent, and A is said to be rank deficient: in particular, if $n = m$, then A is singular.

Equation (1) is relevant to equations (2)and (3) above. If $Ax = 0$ has nonzero solutions, then equation (1) does not have unique solutions[2]. Also, in equation (3), we write $(A - \lambda I)x = 0$ to obtain equation (1) (here I is the identity matrix).

[2] Let $Ax_1 = 0$ with $x_1 \neq 0$, and let $Ax_2 = b$. Then $A(x_1 + x_2) = b$ as well, showing nonuniqueness. Nonunique solutions can disturb, say, an engineer trying to calculate the deflection of a bridge under the weight of a bus.

Equation (1) is needed for understanding when a system is controllable and when it is observable (there are formal definitions of these terms; see [1]). In these applications, the relevant coefficient matrices A should be of full rank, i.e., there should exist no nontrivial solutions for x.

With Numerical Roundoff or Measurement Error

As suggested above, an important property of a matrix is its rank, which equals the number of linearly independent columns it has. For example, if

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

then its rank is 2 (because the third column is the sum of the first two, and so linearly dependent on them).

In numerical work with large matrices that are nearly rank-deficient, the border between invertible and rank-deficient matrices becomes blurred. An important tool here is the singular value decomposition (SVD), which is a characterization of a matrix in terms of some non-negative numbers called its singular values, and some vector directions called its singular vectors. The number of strictly positive singular values equals the rank of the matrix[3].

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[3] Readers interested in a detailed discussion may see, e.g. [2,3]

For a numerical example, consider

$$A = \begin{bmatrix} 1 & -1 & 0.02 \\ 2 & 0 & 1.98 \\ 0 & 4 & 4.01 \end{bmatrix}$$

In exact arithmetic, the rank of A is 3. However, the third column is clearly almost equal to the sum of the first two columns. So the rank is somehow close to 2 (a hazy notion). More concretely, the singular values of A are (from Matlab) $\{5.8962, 2.6898, 0.0164\}$. The third singular value is much smaller than the second.

Suppose that, in an experiment, an engineer is trying to characterize the vibrations of a platform (see *Figure* 1). She has placed several accelerometers on the platform, and measured their output while some machinery on it was running. She may now ask, is the platform

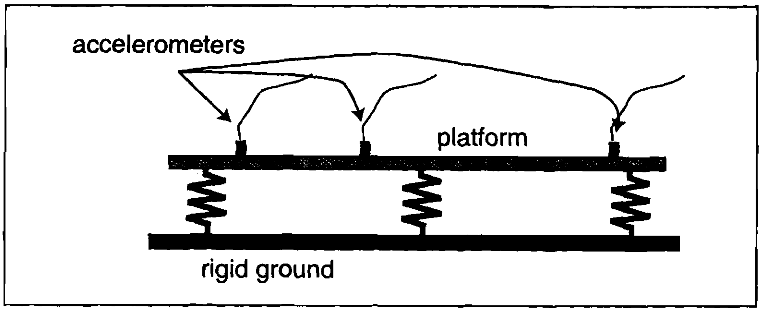

Figure 1. A vibrating platform.

effectively rigid? If not, how many independent types of vibrational motions does it have? These questions, which represent noise-polluted versions of the linear homogeneous system, can be tackled using the SVD.

The Standard Linear System (2)

The equation $Ax = b$ is the backbone of engineering calculations. $Ax = 0$ impinges on it largely to the extent of understanding or eliminating nonuniqueness of solutions.

The commonest situation involves A square and invertible, in which case there is a unique and exact solution which can be found using accurate algorithms that are guaranteed to work. Much ingenuity in engineering has been expended in casting important problems into this form (and, for large systems, storing the matrices and solving the equations iteratively).

The somewhat less common but also important case where A is $m \times n$ with $n < m$ is called an overdetermined system. It is not solvable exactly unless b satisfies special conditions, but can be solved approximately in a *least squares* sense by solving the *normal* equations

$$A^T A x = A^T b,$$

where the T-superscript denotes transpose and $A^T A$ is $n \times n$, i.e., square. In applications, sometimes A is deliberately made overdetermined in order to get a better overall fit for some inexact model, and the resulting equations are solved in a least squares sense as above. In numerical work with roundoff errors, there are nominally equivalent methods that in fact keep accumulating roundoff errors under tighter control [2].

Let us briefly consider an overdetermined system. Consider two numbers p and q. We are told:

1. the sum of the numbers is 6,

2. the second number is twice the first one, and

3. the second number is 3 more than the first one.

The first two conditions above imply $p = 2$, $q = 4$ (provided we ignore the third condition). Similarly, the second and third conditions (ignoring the first) inconsistently imply $p = 3$, $q = 6$. And the first and third condition imply $p = 3/2$, $q = 9/2$. There is no choice of p and q that satisfies all three conditions. Any two of these conditions uniquely determine p and q; and the extra inconsistent condition makes the system overdetermined. To connect with the matrix algebra, we can write the three conditions as

$$p + q = 6, \quad q = 2p, \quad \text{and} \quad q = p + 3,$$

which in matrix notation is

$$\begin{bmatrix} 1 & 1 \\ -2 & 1 \\ -1 & 1 \end{bmatrix} \left\{ \begin{matrix} p \\ q \end{matrix} \right\} = \left\{ \begin{matrix} 6 \\ 0 \\ 3 \end{matrix} \right\}$$

Since the above matrix equation has no solution (being overdetermined), we might use the normal equations

$$\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \left\{ \begin{matrix} p \\ q \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 9 \end{matrix} \right\}$$

whose solution is $p = 1.93$, $q = 4.29$. The reader may check for approximate satisfaction of all three conditions.

We now drop overdetermined systems and focus on square A.

A Boundary Value Problem

Boundary value problems are very common in engineering. They involve differential equations along with boundary conditions. The simplest ones involve second order ordinary differential equations with two boundary points. Here is one such equation:

$$\frac{d^2y}{dx^2} + xy = 1, \quad x \in (0,1), \quad y(0) = y(1) = 0. \qquad (4)$$

A simple way to solve this problem computationally is to choose a large positive integer N, and then use a uniform mesh of points $x_k = kh$, with $k = 0, 1, 2, \quad , N$ and $h = 1/N$ Let $y_k = y(x_k)$. Then we simply write the following equations, where the derivative is approximated using sums and differences so that we eventually get the standard linear system:

$$\frac{y_{k+1} - 2y_k + y_{k-1}}{h^2} + x_k$$

$$\begin{aligned} y_0 &= 0 \quad \text{from boundary conditions,} && (5)\\ y_k &= 1 \quad \text{for } k = 1, 2, \quad N-1, && (6)\\ y_N &= 0 \quad \text{from boundary conditions.} && (7) \end{aligned}$$

In the above equations, all quantities except the y's are known; and so they can be assembled to form (2). This *finite difference* method can be extended to complex problems in 2 and 3 dimensions. It is conceptually simple but in higher dimensional problems requires care near tilted and/or curved boundaries.

An alternative solution technique is to assume an approximate solution of the form

$$y \approx \sum_{i=1}^{N} a_i \phi_i(x) = \sum_{i=1}^{N} a_i \sin i\pi x.$$

Here, as indicated above, the ϕ's are shape functions we choose in advance; and the a's are unknown coefficients that we will find. The specific choice of sines for the ϕ's in this case respects the boundary conditions (this is important). We then proceed by substituting the approximation into

$$\frac{d^2y}{dx^2} + xy - 1$$

to obtain what is called a residual, say $r(x; a_1, a_2, \quad, a_N)$. The next step is called a *Galerkin* projection or the method of *weighted residuals* (see [4]), and involves multiplying the residual by each of the shape functions, integrating over the domain, and setting the result to zero, i.e.,

$$\int_0^1 r(x; a_1, a_2, \quad, a_N) \sin k\pi x \, dx = 0,$$

$$\text{for } k = 1, 2, \quad, N.$$

This again results in a system of the form of equation (2). (These calculations can be conveniently done using symbolic algebra packages like Maple or Mathematica.)

Some numerical results for both the above methods are given in *Figure* 2, for $N = 6$ and $N = 10$. Both methods perform well. The finite difference results are interpolated using broken lines for better visibility alone; only the nodal values (at the corners) are to be used for comparison.

The method of weighted residuals can also be used with shape functions ϕ that are zero everywhere except inside some small regions; this leads to the powerful and

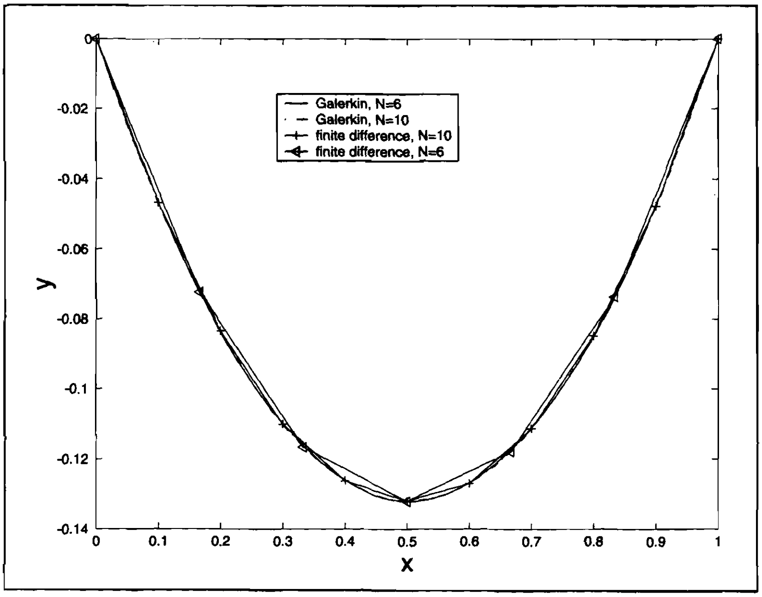

Figure 2. Soution of equation (4).

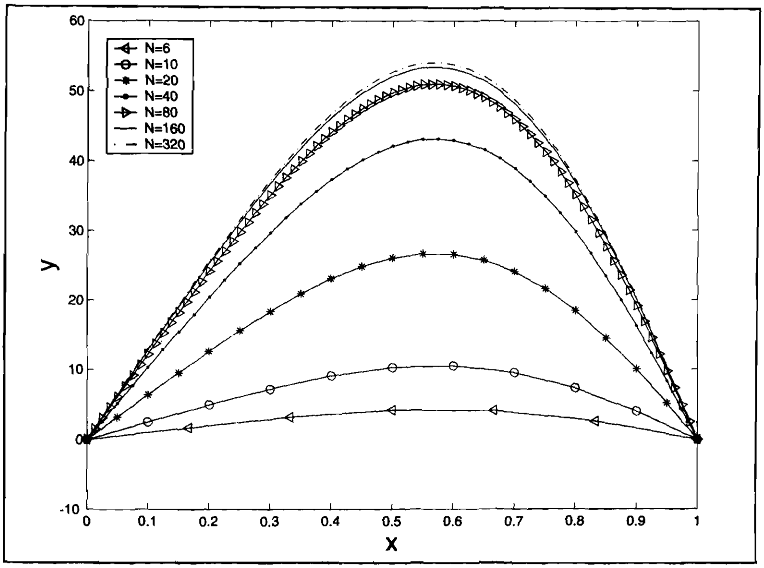

Figure 3. Solution of equation (8).

versatile *finite element* method, well suited for complex geometries, but too specialized for this article.

Signs of Trouble

Let us now consider the similar looking boundary value problem

$$\frac{d^2y}{dx^2} + 19xy = 1, \quad x \in (0,1), \quad y(0) = y(1) = 0. \quad (8)$$

Numerically solving this using the finite difference method as described above, we obtain the results shown in *Figure* 3. Now the solution converges sluggishly, and fairly large N is needed before a reliable solution is obtained. Why? A partial answer lies in the singular values of the coefficient matrix A for each N These values are plotted for $N = 80$, 160 and 320 in *Figure* 4. In each case, it is seen that the smallest few singular values drop off rapidly in magnitude, with the smallest one in each case being much smaller than the second smallest. Although the smallest singular value is nonzero for each N and therefore A is invertible, the numerical shadow of singular A in equation (1) has fallen on the calculation! It is possible to construct special schemes for

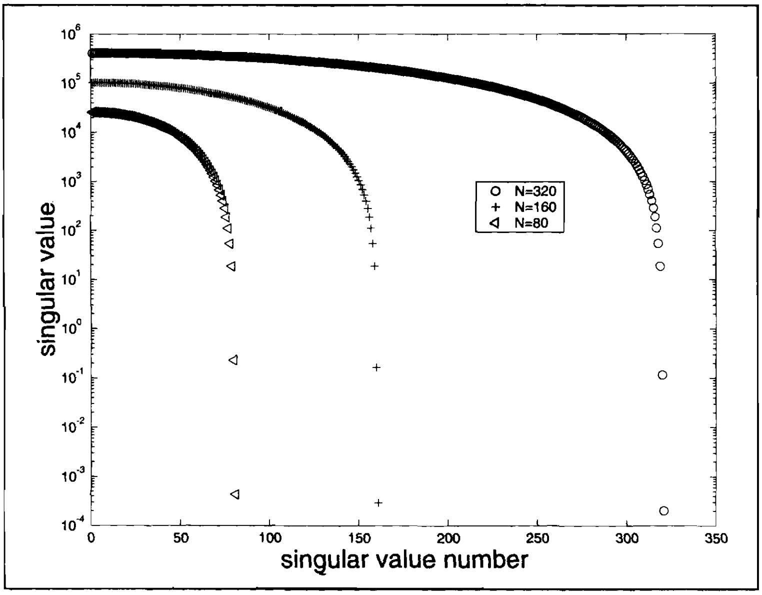

Figure 4. Singular values of A from equation (8) for different N.

tackling near-singular problems, but in practical applications they need to be guided by physical understanding of the problem, and are beyond the scope of this article.

We now proceed to equation 3.

The Linear Eigenvalue Problem (3)

In studying uniqueness of solutions for equations like equation (8), we seek nonzero solutions of

$$\frac{d^2y}{dx^2} + \mu xy = 0, \quad x \in (0,1), \quad y(0) = y(1) = 0. \quad (9)$$

Here, μ is a parameter that is as yet unknown.

The finite difference equations can now be written in the form

$$Ay = \mu By$$

which can in turn be rearranged to

$$A^{-1}By = \frac{1}{\mu}y = \text{ (say) } \lambda y,$$

Eigenvalue problems arise in many places: in linear vibrations, buckling problems, systems of linear differential equations, and stability calculations in a variety of situations, to name a few.

which matches (3). The numerically obtained values of λ (from Matlab) for the case of $N = 10$ are

$$0, 0, 4.19E-4, 9.17E-4, 1.42E-3, 1.92E-3, 2.50E-3,$$

$$3.52E-3, 5.83E-3, 1.27E-2, \text{ and } 5.33E-2,$$

where '$E - 4$' denotes '$\times 10^{-4}$' and so on. The estimate of μ corresponding to $1/(5.33 \times 10^{-2}) = 18.8$, which should be compared with the coefficient in (8). The actual eigenvalue of equation (9) is even closer to 19, as may be found using larger N

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In the discussion of the above three equations (1–3) I have not touched upon the many strategies needed and developed to tackle very large and/or sparse systems, as well as systems with special properties such as A being symmetric and positive definite; these topics, too specialized for this article, are important in applications.

This ends the first part of this article, which has been broken into two for reasons of space. The next part will appear in a later issue. The first section of this article was prompted by conversations with B Gurumoorthy, R N Goverdhan, D Chatterjee and A Ruina read the manuscript and helped me improve it.

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Suggested Reading

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