In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Must Books that are Popular be Closer to Top of the Stack?

Problem: There are n books numbered, say, $1, 2, \quad , n$. These are kept in a stack, one over another, not necessarily in the order of their numbers. Only one book is consulted at a given moment and replaced on top of the stack after consultation. Let p_i = Prob (Book i is consulted at a given moment), $p_i > 0, \sum_{i=1}^{n} p_i = 1$.

Assume that the process is repeated several times, that the repetitions are independent of each other, and that p_i do not change. Can you prove that, on the average, the more popular books are likely to be closer to the top of the stack? Mathematically, if d_i is the expected depth of Book i from the top of the stack, you are required to prove that $d_i \leq d_j$ if $p_i \geq p_j$. (Here depth of a book merely gives its position from the top of the stack.)

Solution (Proof): Define random variables Y_{kj} as follows:

$$Y_{kj} = \begin{cases} 1, & \text{if Book } k \text{ is above Book } j, \\ 0, & \text{otherwise.} \end{cases}$$

Let D_j = Depth of Book j at a given moment.

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Random variable, expectation.

Obviously, $d_j = E(D_j)$, where E denotes mathematical expectation. Evidently,

$$D_j = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} Y_{kj}.$$

On the average, the more popular books are likely to be closer to the top of the stack.

Define $p_{kj} = P$ (Book k is above Book j), $k \neq j$.

We have,

$$d_j = E(D_j) = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} E(Y_{kj}) = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} p_{kj}, \qquad (1)$$

because $E(Y_{kj}) = 1 \quad P(Y_{kj} = 1) + 0 \quad P(Y_{kj} = 0) = p_{kj}$.

Now, to obtain an expression for p_{kj} we proceed as follows.

We argue that there are two mutually exclusive ways in which Book k can be over Book j:

(i) Book k was already over Book j (probability p_{kj}) and Book j is **not** consulted at the moment (probability $1 - p_j$). The compound event has probability $p_{kj}(1 - p_j)$ due to independence.

(ii) Book j was over Book k (probability p_{jk}) and Book k is consulted at the moment (probability p_k) so that it goes on top of the stack after consultation and hence automatically over Book j as well! The compound event has probability $p_{jk}p_k$, due to independence.

Applying the law of total probability on mutually exclusive cases (i) and (ii), we have

$$p_{kj} = p_{kj}(1 - p_j) + p_{jk}p_k.$$

But $p_{kj} + p_{jk} = 1$ (since the events of Book k being over Book j and vice versa are complementary).

Therefore,

$$p_{kj} = p_{kj}(1 - p_j) + (1 - p_{kj})p_k.$$

After simplification, one gets the desired expression

$$p_{kj} = \frac{p_k}{p_j + p_k}. \tag{2}$$

Using (2) in (1), we have

$$d_j = 1 + \sum_{\substack{k=1 \\ k \neq j}}^{n} \left(\frac{p_k}{p_j + p_k} \right)$$

Replacing j by i,

$$d_i = 1 + \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(\frac{p_k}{p_i + p_k} \right)$$

Comparing the expressions of d_i and d_j, one observes that if $p_i \geq p_j$, each term in the sum for d_i would be less than or equal to the corresponding term in the sum for d_j, leading to $d_i \leq d_j$, thereby completing the proof.

Exercise: Prove that at a randomly chosen moment, the probability for each book to be in its proper place is given by the expression

$$p_1 \left(\frac{p_2}{1 - p_1} \right) \left(\frac{p_3}{1 - p_1 - p_2} \right) \quad \left(\frac{p_n}{1 - p_1 - p_2 \quad - p_{n-1}} \right)$$

"Space isn't remote at all. It's only an hour's drive away if your car could go straight upwards."

– Sir Fred Hoyle