Graph Theory to Pure Mathematics: Some Illustrative Examples

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1. Introduction

Mathematicians are generally aware of the significance of graph theory as applied to other areas of science and even to societal problems. These areas include organic chemistry, solid state physics, statistical mechanics, electrical engineering (communication networks and coding theory), computer science (algorithms and computation), optimization theory, and operations research. The wide scope of these and other applications has been well documented [1,2].

However, not everyone realizes that the powerful combinatorial methods found in graph theory have also been used to prove significant and well-known results in a variety of areas of pure mathematics. Perhaps, the best known of these methods are related to a part of graph theory called *matching theory*. For example, results from this area can be used to prove Dilworth's chain decomposition theorem for finite partially ordered sets. A well-known application of matching in *group theory* shows that there is a common set of left and right coset representatives of a subgroup in a finite group. Also, the existence of matchings in certain infinite bipartite graphs played an important role in Laczkovich's affirmative answer to Tarski's 1925 problem of whether a circle is piecewise congruent to a square. Other applications of graph theory to pure mathematics may be found scattered throughout the literature.

Recently, a collection of examples [3], showing the application of matching theory, is applied to give a very simple constructive proof of the existence of Haar measure on compact topological groups. However, the other combinatorial applications of [3] do not focus on graph theory. The graph-theoretic applications presented in this article do not overlap with those in [3] and no attempt has been made at a survey. Rather, we present some examples, whose statements are well known or are easily understood by mathematicians who are not experts in the area. The definitions and proofs are explained in a relatively short space, without much technical detail. The proofs exhibit the elegance of graph-theoretic methods, although, in some cases, one must consult the literature in order to complete the proof.

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2. Preliminaries

An undirected graph $G = (V, E)$ is a pair in which V is a set, called the vertices of G, and E is a set of 2-element subsets of V called the edges of G. An edge $e \in E$ is denoted by $e = xy$, where x and y are the end vertices of e. The degree of a vertex v, $deg(v)$, is the number of edges incident with v.

A *trail* of length n in a graph G is a sequence of vertices $x_0, x_1, \ldots, x_n$ ($x_i \in V$), such that for $i = 0, 1, \ldots, n-1$, $x_i x_{i+1}$ is an edge of G, and further, all edges $x_i x_{i+1}$ are distinct. If $x_0 = x_n$, then the trail is said to be closed. When all the vertices in the sequence are distinct, the trail is called a *path*. A closed trail, all of whose vertices are distinct except for x_0 and x_n, is called a *cycle*.

A graph G is *connected* if any two vertices of G are joined by a path in G. Otherwise, G is said to be disconnected. The components of G are the maximal connected subgraphs of G. A tree is a connected graph without cycles. A graph $G = (V, E)$ is said to be bipartite if V can be partitioned into two non-empty subsets A and B such that each edge of G has one end vertex in A and one end vertex in B. Then, G is also denoted by $G = (A, B; E)$.

3. Cantor–Schröder–Bernstein Theorem

The following theorem was stated by Cantor who did not give a proof. The theorem was proved independently by Schröder(1896) and Bernstein (1905). The ideas behind the proof presented here can also be found in [4].

Theorem 3.1.(Cantor–Shröder–Bernstein): *Let A and B be two sets. If there is an injective mapping $f : A \rightarrow B$ and an injective mapping $g : B \rightarrow A$, then there is a bijection between A and B.*

Proof: Without loss of generality, we may assume that $A \cap B = \phi$. Define a bipartite graph $G = (A, B; E)$, where $xy \in E$ if and only if either $f(x) = y$ or $g(y) = x$, for $x \in A$ and $y \in B$. By our hypothesis, $1 \leq deg(v) \leq 2$ for each vertex v of G. Therefore, each component of G is either a one-way infinite path (i.e., a path of the form $x_0, x_1, \ldots, x_n$), or a cycle of even length with more than two vertices, or an edge. Note that a finite path of length at least 2 cannot be a component of G. Hence, there is in each component, a set of edges such that each vertex in the component is incident with precisely one of these edges. Hence, in each component, the subset of vertices from A is of the same cardinality as the subset of vertices from B. □

4. Fermat's (Little) Theorem

There are many proofs of Fermat's Little Theorem, that include even short algebraic or number theoretic proofs. The first known proof of the theorem was given by Euler in his letter to Goldbach, dated 6th March, 1742. The idea of the graph-theoretic proof presented below can be found in [5], where this method, together with some number-theoretic results, was used to prove Euler's generalization to non-prime modulus.

The first known proof of Fermat's Little theorem was given by Euler in his letter to Goldbach, dated 6th March, 1742.

Theorem 4.1.(Fermat): *Let p be a prime such that a is not divisible by p. Then a^p-a is divisible by p.*

Proof: Consider the graph $G = (V, E)$, where V is the set of all sequences $(a_1, \ldots, a_p)$ of natural numbers between 1 and a (inclusive), with $a_i \neq a_j$ for some

$i \neq j$. Clearly, V has $a^p - a$ elements. For any $u \in V$ $u = (u_1, \ \ , u_p)$, let us say that $uv \in E$ if and only if $v = (u_p, u_1, \ \ , u_{p-1})$. Clearly, each vertex of G is of degree 2; hence, each component of G is a cycle of length p. But then, the number of components must be $\frac{(a^p - a)}{p}$. Therefore, $p|(a^p - a)$. □

5. Existence of a Non-Measurable Set

The following proof of the existence of a subset of the real numbers $\mathbb{R}$, which is non-measurable in the Lebesgue sense, is due to Thomas [6]. He wrote his paper while he was an undergraduate student. We realize that many readers may still prefer Vitali's proof. However, it is quite unexpected that this theorem can be reduced to the theorem below, an easily proved result in measure theory, by using only discrete mathematics.

A simple well-known result from graph theory says that a graph $G = (V, E)$ is bipartite if and only if all its cycles are of even length. Consider now the graph $T = (\mathbb{R}, E)$, where $xy \in E$ if and only if $|x - y| = 3^k$, with $k \in \mathbb{Z}$. In order to show that T is bipartite, suppose that $x_0, x_1, \ \ , x_{n-1}$, where $x_n = x_0$, is a cycle of length n in T Then, by the definition of T

$$x_n = x_{n-1} \pm 3^{k_n} = x_{n-2} \pm 3^{k_{n-1}} \pm 3^{k_n} =$$

$$= x_0 \pm 3^{k_1} \pm 3^{k_2} \pm \quad \pm 3^{k_n}$$

and thus, $\pm 3^{k_1} \pm 3^{k_2} \pm \quad \pm 3^{k_n} = 0$, where $\{k_i\}_{i=1}^n$ is a set of integers. Multiplying both sides by 3^N where N is an integer such that $N + k_i > 0$ for all $1 \leq i \leq n$, yields $\pm 3^{N+k_1} \pm 3^{N+k_2} \pm \quad \pm 3^{N+k_n} = 0$, which implies that n is even, since otherwise the left side of the above equation is odd, a contradiction. Thus, T is bipartite.

Hence, there are sets A and B with $A \cap B = \phi$, $A \cup B = \mathbb{R}$ such that each edge of T is incident with one vertex in A and the other vertex in B. If both A and B were measurable, then at least one of them, say A, would have positive measure. Furthermore, for each integer k, $A + 3^k \leq B$, which yields $A \cap (A + 3^k) = \phi$. Since $3^k \to 0$ as $k \to -\infty$, this contradicts the following theorem, which is a standard result in measure theory. For convenience of the reader, we include the proof from [6].

Theorem 5.1. *Let M be a set of real numbers with positive Lebesgue measure. Then, there exists a $\delta > 0$ such that for every $x \in \mathbb{R}$, $|x| < \delta$, $M \cap (M + x) \neq \phi$.*

Proof: Find a closed set F and an open set G with $F \leq M$ and $F \subset M$ such that $3\lambda(G) < 4\lambda(F)$ (where $\lambda(\cdot)$ is the Lebesgue measure). Since G is a countable union of disjoint open intervals, there is one among them, say, I, such that $3\lambda(I) < 4\lambda(F \cap I)$. Let $\delta = \frac{1}{2}\lambda(I)$ and suppose that $|x| < \delta$. Then, $I \cup (x + I)$ is an interval of length less than $\frac{3}{2}\lambda(I)$, which contains both $F \cap I$ and $x + (F \cap I)$. The last two sets cannot be disjoint, since otherwise we have

$$\frac{3}{2}\lambda(I) = \frac{3}{4}\lambda(I) + \frac{3}{4}\lambda(I) < \lambda[(F \cap I) \cup (x + (F \cap I))]$$

$$\leq \lambda(I \cup (x + I)) \leq \frac{3}{2}\lambda(I),$$

which is a contradiction. Hence, $\phi \neq (F \cap I) \cap (x + (F \cap I)) \leq M \cap (x + M)$, completing the proof. □

Remark 5.1. *It is well known that a non-measurable set cannot be constructed without using the axiom of choice. The graph T above is constructed without using the axiom of choice. Note that T is not connected, and in fact, each component of T has only a countable number of vertices. Thus, to define A and B we need to invoke the axiom of choice.*

6. Brouwer's Fixed-Point Theorem

Brouwer's fixed point theorem states that every continuous mapping f of a closed n-disc onto itself has a fixed point. This theorem is an easy consequence of a simple combinatorial lemma due to Sperner (1928). Sperner's lemma concerns the decomposition of a simplex, such as a line segment, triangle, tetrahedron, and so on, into smaller simplices. For the sake of simplicity, consider the two-dimensional case. Let T be a closed triangle in the plane. A subdivision of T into a finite number of smaller triangles is said to be a simplicial if any two intersecting triangles have either a vertex or a whole side in common. Suppose that a simplicial subdivision of T is given. Then, a labeling of the vertices of triangles in the subdivision in three symbols $0, 1, 2$ is said to be proper if

(i) The three vertices of T are labeled 0, 1, and 2 (in any order) and,

(ii) For $0 \leq i < j \leq 2$, each vertex on the side of T joining vertices labeled i and j is labeled either i or j.

We call a triangle in the subdivision whose vertices receive all three labels a *distinguished* triangle.

Lemma 6.1. (Sperner): *Every properly labeled simplicial subdivision of a triangle has an odd number of distinguished triangles.*

Proof: Since a closed 2-disc is homeomorphic to a closed triangle, it suffices to prove that a continuous mapping of a closed triangle to itself has a fixed point. Let f be any continuous mapping of T to itself, and suppose that $f(a_0, a_1, a_2) = (a'_0, a'_1, a'_2)$, where T is a given closed triangle with vertices x_0, x_1, and x_2. Hence, any $x \in T$ is $x = \sum_{i=0}^{2} a_i x_i$, with $a_i \geq 0$ and $\sum_{i=0}^{2} a_i = 1$. Define

$$S_i = \{(a_0, a_1, a_2) : (a_0, a_1, a_2) \in T, a'_i \leq a_i\}.$$

To show that f has a fixed point, it is enough to show that $\cap_{i=0}^{2} S_i \neq \phi$. For suppose that $(a_0, a_1, a_2) \in \cap_{i=0}^{2} S_i$. Then, by definition of S_i, we have that $a'_i \leq a_i$, $\forall i$, and

Yet, no matter how we partition {1,2,...,14} into three subsets, there always exists a subset of the partition which contains a solution to the equation $x+y=z$.

$\sum a_i' = \sum a_i$. Hence, $(a_0', a_1', a_2') = (a_0, a_1, a_2)$, that is, (a_0, a_1, a_2) is a fixed point of f. Therefore, consider an arbitrary subdivision of T and a proper labeling such that each vertex labeled i belongs to S_i. It follows from Sperner's lemma that there is a triangle in the subdivision whose three vertices belong to S_0, S_1, and S_2. Now this holds for any subdivision of T and since it is possible to choose subdivisions in which each of the smaller triangles are of arbitrarily small diameter, we conclude that there exists three points of S_0, S_1, and S_2 which are arbitrarily close to one another. Because the sets S_i are closed, one may deduce that $\cap_{i=0}^{2} S_i \neq \phi$. For details of the above proof and other applications of Sperner's lemma, the reader is referred to [7]. □

7. Schur's Theorem

Consider a partition $(\{1,4,10,13\}, \{2,3,11,12\}, \{5,6,7,8,9\})$ of the set of integers $\{1, 2, \quad 13\}$. We observe that in no subset of the partition are there integers x, y, and z (not necessarily distinct) which satisfy the equation $x + y = z$. Yet, no matter how we partition $\{1, 2, \quad 14\}$ into three subsets, there always exists a subset of the partition which contains a solution to the equation $x + y = z$. Schur (1916) proved that "In general, given any positive integer n, there exists an integer f_n such that, in any partition of $\{1, \quad f_n\}$ into n subsets, there is a subset, which contains a solution to $x + y = z$" Schur's theorem follows from the existence of the Ramsey numbers r_n. By a complete graph K_n, we mean the graph on n vertices in which any two vertices are adjacent. A k-edge coloring of a graph G is an assignment of k colors $1, \quad k$ to the edges of G. A k-edge coloring is called proper if no two adjacent edges have the same color. Ramsey (1930) showed that, given any two positive integers k and ℓ, there exists a smallest integer $r(k,\ell)$ such that every graph on $r(k,\ell)$ vertices contains either a complete graph on k vertices or an independent set of ℓ vertices[1]. The following theorem of

A set of vertices in a graph is called *independent* if no two of them are adjacent.

Ramsey is due to Erdös and Szekeres (1935) and Greenwood and Gleason (1955).

Theorem 7.1 (Ramsey): *For any two integers* $k \geq 2$, $\ell \geq 2$, $r(k, \ell) \leq r(k, \ell - 1) + r(k-1, \ell)$. *If both the terms on the right hand side are even then strict inequality holds.*

Ramsey numbers have a natural generalization. Define $r(k_1, \quad k_m)$ to be the smallest integer n such that every m-edge coloring $(E_1, \quad , E_m)$ of K_n contains for some i, a complete subgraph on k_i vertices, all of whose edges are of color i. Let r_n denote the Ramsey number $r(k_1, \quad k_n)$ with $k_i = 3$ for all i. Then it is easy to show that $r_n \leq n(r_{n-1} - 1) + 2$ and $r_n \leq \lfloor n!e \rfloor + 1$.

Thoerem 7.2. *Let* $(S_1, \quad , S_n)$ *be any partition of the set of integers* $\{1, \quad , r_n\}$. *Then, for some* i, S_i *contains three integers* x, y, *and* z *such that* $x + y = z$.

Proof: Consider the complete graph K_{r_n}. Color the edges of this graph in colors $1, \quad , n$ by the rule that the edge uv is assigned the color i if $|u - v| \in S_i$. By Ramsey's theorem there exists a monochromatic triangle: that is, there are three vertices a, b, and c such that ab, bc, and ca have the same color, say i. Assume that $a > b > c$ and write $x = a - b$, $y = b - c$, $z = c - a$. Then $x, y, z \in S_i$ and $x + y = z$. □

Remark 7.1. *Let* s_n *denote the least integer such that, in any partition of* $\{1, \quad , s_n\}$ *into* n *subsets, there is a subset which contains a solution to* $x + y = z$. *It is easily seen that* $s_1 = 2$, $s_2 = 5$, $s_3 = 14$. *Further from the above theorem we have the upper bound* $s_n \leq r_n \leq \lfloor n!e \rfloor + 1$.

8. Universal Group Graph on $\mathbb{Z}$

In trying to describe graphs by integers we may ask:

Is there a set T *of integers with the following property: The vertices of each finite graph can be labeled by differ-*

A graph is a group-graph if and only if it is non-empty and its automorphism group contains a subgroup acting strictly transitively on the vertices.

ent numbers such that two arbitrary vertices are adjacent if and only if the difference of their labels belongs to T ?

The above question can be formulated in a more concise way as follows:

Theorem 8.1(Characterization Theorem): *There is a group graph on $\mathbb{Z}$ which contains all finite graphs as edge induced subgraphs up to isomorphism.*

Algebraists conceive a graph as a pair (V, ∇), where V is a set and ∇ is an irreflexive, symmetric relation on V The elements of V are vertices and ∇ represents the adjacency. A group-graph (H, ρ_T) on H is defined as follows: Let H be a group and let T be a subset of H such that $1 \notin T$, $T^{-1} \subseteq T$ The relation ρ_T is defined by $x\rho_T y$, if $xy^{-1} \in T$ for all $x, y \in H$ Obviously, ρ_T is irreflexive and symmetric. A group graph is not just regular but even homogeneous in the sense that its automorphism group acts transitively on the vertices, since the right translations of the group are automorphisms of the group graph. But the converse is not necessarily true. That is, not every homogeneous graph is a group graph. For example, it is easy to see that the five regular polyhedron graphs are homogeneous, but the dodecahedron graph is not a group graph. In view of this, a group graph can be defined alternatively as: A graph is a group-graph if and only if it is non-empty and its automorphism group contains a subgroup acting strictly transitively on the vertices. Even as simple a case as the cyclic group of order six produces several interesting examples of group graphs. That is, defining the addition modulo 6 on $\mathbb{Z}_6 = \{0, 1, \quad 5\}$ we get the following group graphs $(\mathbb{Z}_6, \rho_T)$. (For more on this topic refer to [2,8].)

$T = \phi$ 6 vertices	$T = \{3\}$ 3 edges	$T = \{2, 4\}$ 2 triangles	$T = \{1, 5\}$ a hexagon	$T = \{2. 3. 4\}$ prism
$T = \{1, 3, 5\}$ complete bipartite graph $K_{3,3}$	$T = \{1, 2, 4, 5\}$ octahedron	$T = \{1, 2, 3, 4, 5\}$ complete graph K_6		

Suggested Reading

[1] L Caccetta and K Vijayan, Applications of graph theory,in *Fourteenth Australasian Conference on Combinatorial Mathematics and Computing*, (Dunedin, 1986); Ars Combin., Vol. 23, pp. 21-77, 1987.

[2] F S Roberts, Graph Theory and its Applications to the Problems of Society, *CBMS - NSF Monograph* 29, SIAM Publications, Philadelphia, 1978.

[3] L Lovasz, L Pyber, D J A Welsh, and G M Ziegler, Combinatorics in pure mathematics, in *Handbook of Combinatorics*, Editors: R L Graham, M Grotschel and L Lovasz, Elsevier Science BV, Amsterdam, 1996.

[4] D Konig, Theorie der endlichen und unendlichen Graphen, *Akademische Verlagsgesellschaft*, Leipzig, 1936; reprinted by Chelsea, New York, 1950.

[5] K Heinrich and P Horak, Euler's Theorem, *American Mathematical Monthly*, Vol.101, p. 260, 1994.

[6] R Thomas, A combinatorial construction of a non-measurable set, *American Mathematical Monthly*, Vol. 92, pp. 421-422, 1985.

[7] C B Tompkins, Sperner's lemma and some extensions, in *Applied Combinatorial Mathematics*, Editor: E F Beckenbach, Wiley, New York, ch.15, pp.416-455, 1964.

[8] R Schnabel and F R G Kiel, *Topics in Combinatorics and Graph Theory*, Editors: Rainer Bodeniek and Rudolf Henn, Physica-Verlag-Heidelberg, 1990.

[9] C C Chen and H H Teh, Embedding theorems of homogeneous graphs, *Bull. Mathematical Society*, Nanyang University, pp.21-30, 1965.

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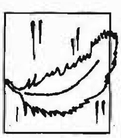

"I have often said from the podium that although it is gravity that holds my feet to the ground, it is the electromagnetic force that stops me from falling through the ground. Electromagnetism binds the atoms together and puts a solid floor beneath my feet."

Sheldon L Glashow
(Higgins Professor of Physics at Harvard University; Nobel Laureate 1979)
Interactions, Warner Books, New York 1988.