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Must Stacked Cards Topple if They Overhang ?

How high can one stack several playing cards, one over another, with the top card completely overhanging the bottom one but still not toppling over? More precisely, how much can the top card overhang the bottom one without toppling over? This intriguing puzzle has the surprising answer that one could have as much overhanging as one desires provided the number of cards is large enough! See also [1].

Consider the longer edges of the cards to be along the same direction and the cards to be overhanging to the right, say. Let the longer side of each card have length x. Assume that the number of cards is n and that the center of gravity of the top i cards is exactly in line with the right edge of the $i+1$-th card from the top. This is the extreme case such that the cards just manage not to topple over. In this case, it is a nice (but not easy) exercise to show that the overhang of the i-th card over the $i+1$-th one is $\frac{x}{2i}$. The reader is encouraged to supply a proof; it is similar to the proof we give for the continuous version of the problem later.

Therefore, the maximal possible distance by which the top card can overhang the bottom one without toppling over, is given by the sum

$$\frac{x}{2}+\frac{x}{4}+\frac{x}{6}+\quad+\frac{x}{2(n-1)}.$$

Since this series diverges with n, this means that one can have overhanging of the topmost card over the bottom one by as much as one wants without toppling, provided n is sufficiently large.

Let us discuss a more general (continuous) version of the problem. To fix normalisation, we assume that the length of each card is 2 units of the X-axis, and that the

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bottom card has its center at the origin. We also denote by $f(t)$, the position of the center of (gravity of) the card at height t. Let T be the total height of the stack, and, as before, we think of cards as overhanging to the right (the positive side of X-axis). We normalise the thickness (height) of each card so that we have $f(t) = 0$ for $t = 1$. The condition that a card avoids toppling over is that, for each t, the center of gravity of the cards from height t to T is not to the right of the right edge of the card at height t. Thus, we must have

$$\frac{1}{T-t}\int_t^T f(i)di \leq f(t) + 1.$$

This follows from the property that the position of the center of gravity of the cards from height t to T (these cards constitute a height of $(T-t)$ units on the Y-axis) will correspond to the arithmetic mean of the corresponding positions. Note that the 1 on the right hand side of the inequality comes from the half-length of the card since $f(t)+1$ is the position of the right end of the card at height t. When the overhanging is the maximum possible, then the inequality becomes an equality, i.e.

$$\frac{1}{T-t}\int_t^T f(i)di = f(t) + 1. \tag{1}$$

We write $f(y) = F'(y)$ and

$$\int_t^T f(i)di = F(T) - F(t).$$

Differentiating (1) with respect to t gives

$$\frac{1}{(T-t)^2}\int_t^T f(i)di + \frac{1}{T-t}\frac{d[F(T)-F(t)]}{dt} = \frac{df(t)}{dt}.$$

Using (1) again, this simplifies to

$$\frac{1}{T-t} = df(t)/dt. \tag{2}$$

Integrating (2), we obtain

$$-\log(T-t)+D=f(t),$$

where D is a constant arising due to integration.

For the card at the bottom, $f(t)=0$ and $t=1$. This leads to the final solution

$$f(t)=\log\frac{T-1}{T-t}.$$

For any function which grows larger than this function $f(t)$, the cards will topple. Since $f(T)$ is infinite, we can go as far as we like.

In order to make the above mathematics (which is correct) practically viable, one may have to make some physical assumptions like the following:

(a) Cards should be made by material as inelastic as possible so that the deformation of the cards(especially at the edges) due to loading effect is minimised,

(b) There is no external effect like earth vibration which can transmit to the top of the stack; this increases the transverse vibration at the top which can deflect the point of center of gravity, and

(c) There is no wind loading effect.

To achieve (b) and (c), perhaps the experiment has to be conducted in vacuum.

Suggested Reading

[1] Harry Zaremba, *Journal : Technology Review*, Vol. 75, No. 2, pp.66-67, 1972.