

Wavelet Transforms: Application to Data Analysis – II

Jatan K Modi, Sachin P Nanavati, Amit S Phadke and Prasanta K Panigrahi



Jatan K Modi (top left) is currently a visiting faculty at DDIT, Gujarat. His areas of interest are artificial intelligence, compilers and image processing.

Sachin P Nanavati (top right) has joined the Scientific and Engineering Computing Group of C-DAC, Pune. He enjoys trekking and bird watching.

Amit S Phadke (bottom left) is presently working in 'Cytotec', a software firm based in Vadodara, dealing with image processing and 'Flow Cytometry' software. His research interests include image processing and artificial intelligence.

Prasanta K Panigrahi (bottom right) is currently with the Quantum Information and Quantum Optics Division at PRL. His current research interests are in the areas of quantum information, solitons and wavelets.

We illustrate here, the use of wavelets both in one and two dimensions. We outline the procedure to construct wavelet transform in two dimensions. The method of forward transform, reconstruction and the various steps involved are explained pictorially.

Introduction

In the first part¹ of this article, we showed how to carry out discrete wavelet transform of a data set (see also [1]). The basis sets comprising father, mother and daughter wavelets make up an ideal family, which systematically separates out any data into its averaged part and variations at different scales. Different wavelets differ in the manner they take these averages and differences. The simplest Haar wavelet employs straightforward addition and subtraction of two members at a time, whereas more sophisticated wavelets use weighted averages and differences, involving more than two members of the data sets. The data comprising 2^N members can be broken up into N -levels, the last one being the low-pass or average coefficient. The other members are called high-pass coefficients. The level-1 high-pass coefficient is half the size of the data and keeps track of the variations at the highest resolution. The higher level high-pass coefficients locate variations at lower resolutions.

Getting Started in the Wavelet Domain

Mathematically a data set, written here as a function $f(t)$ for convenience, can be represented in wavelet domain as:



$$f(t) = \sum_{k=-\infty}^{\infty} c(k) \phi_k(t) + \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} d_j(k) \psi_{j,k}(t). \quad (1)$$

Here,

$$c(k) = \int_{-\infty}^{\infty} \phi_k^*(t) f(t) dt, \quad \text{and}$$

$$d_j(k) = \int_{-\infty}^{\infty} \psi_{j,k}^*(t) f(t) dt \quad (2)$$

are respectively, the low-pass and high-pass coefficients.

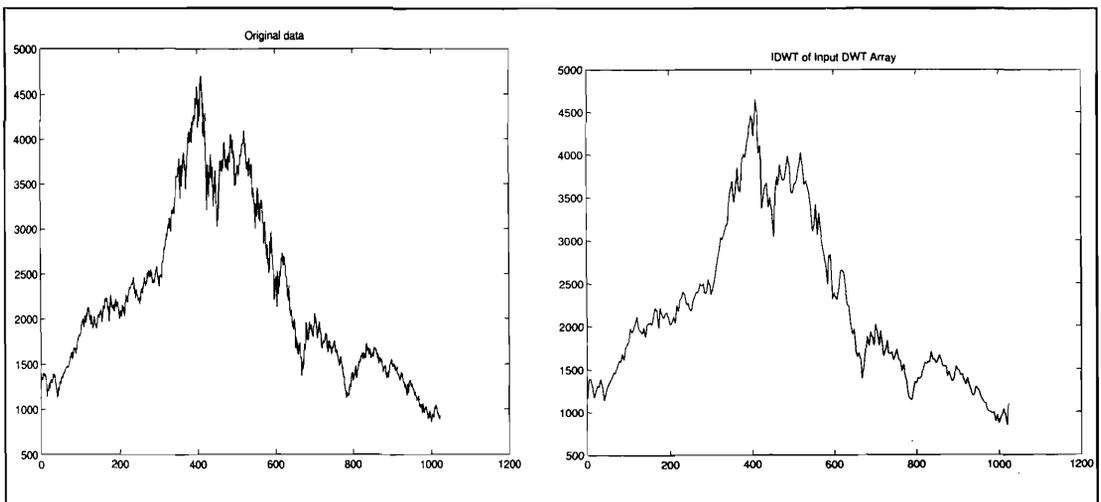
One very useful application of wavelets, particularly for the experimentalists, is in the smoothening of the data. Since the wavelets systematically separate out variations at different scales, one can throw away the small coefficients, representing smaller variations, at any scale of one's choice. This procedure is called 'thresholding'. The reconstructed data will retain large variations and will appear smooth. In *Figure 1* we have shown the original data and the reconstruction of the same after thresholding, to illustrate the above point. There are many ways of thresholding, for example one can remove level 1 detailed coefficients and carry out the reconstruction. In this case, the finest variations in the data set can

¹ Wavelet transforms: Applications to Data Analysis-I, *Resonance*, Vol.9, No.11, pp.10-22, 2004.

Keywords.

Discrete wavelet transform, multi-resolution analysis (MRA).

Figure 1. Original data (1024 data points) and inverse discrete wavelet transform (IDWT) after removal of small coefficients through thresholding. Notice the smooth nature of the reconstructed figure.



can be removed. The readers are referred to Donoho's article [1] for more details.

Another application of wavelet is to identify characteristic variations in the data set. For that purpose, one first plots the power spectrum. As defined in Part 1 of this article, the power or energy at a given level is the sum of the square of the wavelet coefficients at that level. If a characteristic variation is present at a certain scale, the corresponding wavelet coefficients will be large, giving a high value of power at that scale. We show the power spectrum of the data set in *Figure 2* to bring out this point. One notices that the fourth level power is high.

The extension of wavelet transform to two dimensions is rather straightforward. It is similar to the Fourier transform in the sense that, the direct product of two, one dimensional transforms is employed for carrying out the two dimensional transform. As is obvious, two dimensional wavelet transform operates on a two-dimensional (2-D) data set, which is nothing but a matrix.

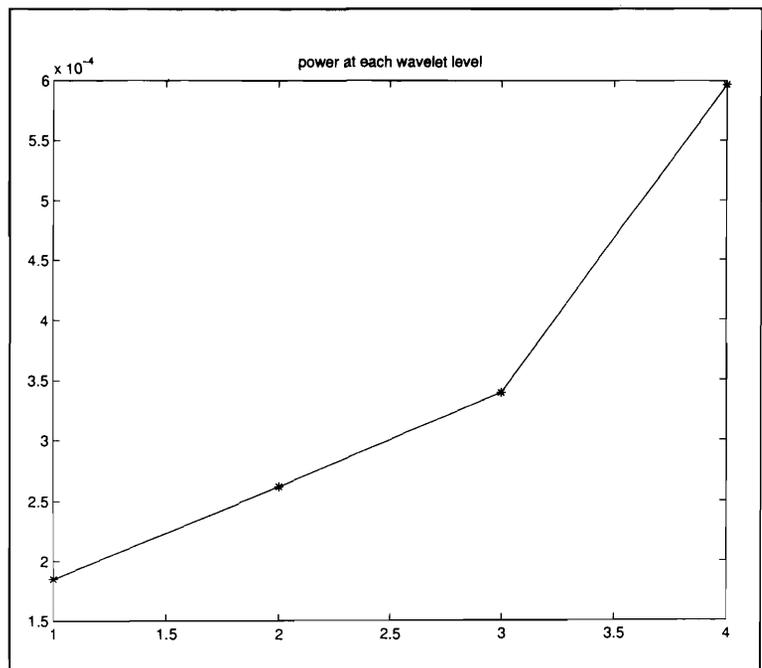


Figure 2. Power captured at each level (only high-pass coefficients. Notice the high value of power at the fourth level.)

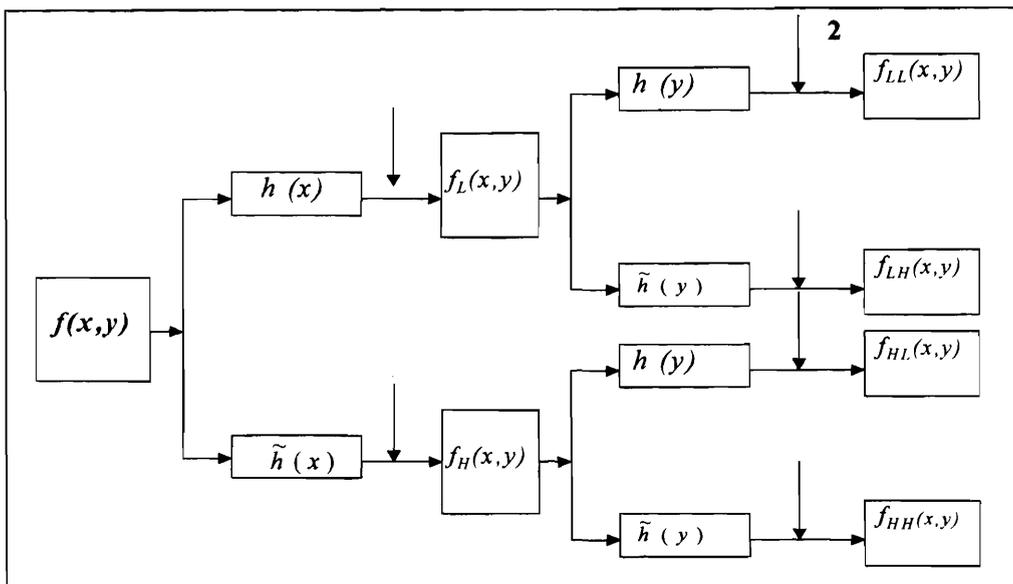
Since our familiar images are two dimensional arrays of numbers in their digital representation, we use a matrix for illustrating the principles involved in 2-D wavelet transform.

2-Dimensional Forward Wavelet Transform

The 2-D forward wavelet transform can be accomplished by two separate 1-D transforms. We will first explain the general algorithm for carrying out the transform over matrices or images, which represent a two dimensional data set. Subsequently, we will explicitly see what happens when a matrix is transformed by using Haar wavelets.

As shown in *Figure 3*, a $M \times N$ image $f(x, y)$ is first filtered along the x dimension, using low-pass $h(x)$, resulting in a low-pass image $f_L(x, y)$. After down-sampling by 2 along x -axis, the columns of $f_L(x, y)$ are now reduced by half, *i.e.*, the size is now $M \times (N/2)$. f_L is now filtered along the y dimension, with both $h(y)$ and $\tilde{h}(y)$, resulting in $f_{LL}(x, y)$ and $f_{LH}(x, y)$. Both are of $(M/2) \times (N/2)$ sizes. We have taken M and N to be even integers. The above procedure is then repeated for

Figure 3. 1-level, 2-D forward discrete wavelet transform (DWT). The down-arrow denotes down-sampling by 2. Here, the original matrix $f(x,y)$ is decomposed into four components: $f_{LL}(x,y)$ which denotes the low-pass (or average coefficients) and $f_{LH}(x,y)$, $f_{HL}(x,y)$ and $f_{HH}(x,y)$ denoting the high-passes (or detail coefficients). h and \tilde{h} denotes low pass and high-pass filter coefficients, respectively.



Suggested Reading

- [1] D L Donoho, De-noising by soft thresholding, *IEEE Trans. Inform. Theory*, Vol. 41, pp. 613-627, 1995.
- [2] S Nanavati and P Panigrahi, Wavelet transform: A new mathematical microscope, *Resonance*, Vol.9, No.3, pp.50-64, March 2004.
- [3] R C Gonzalez and R E Woods, *Digital Image Processing*, Pearson Education Inc., Delhi, 2003.
- [4] S Nanavati and P Panigrahi, Wavelets: Applications to Image Compression, to appear in *Resonance*.

$f(x, y)$ using high-pass $\tilde{h}(x)$ which results in $f_{HL}(x, y)$ and $f_{HH}(x, y)$. For the purpose of illustration, consider a two-dimensional data set having four entries,

$$f(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Using Haar filters we get,

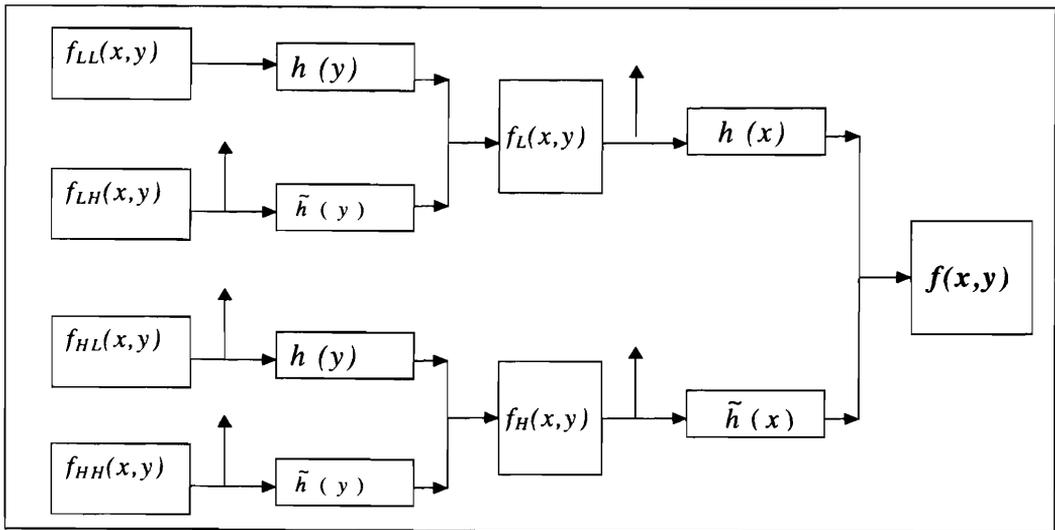
- $f_{LL}(x, y) = \frac{1}{\sqrt{2}} \left[\left(\frac{a+b}{\sqrt{2}} \right) + \left(\frac{c+d}{\sqrt{2}} \right) \right]$. This is called the *average part*. Note that the first subscript L of f_{LL} indicates that the elements of $f(x, y)$ have been filtered with *lowpass* filters along x dimension for all the rows (hence '+' sign in the fractions), while the next L says that the downsampled f_L has been subjected to *lowpass* operation along y dimension (hence '+' sign between the two fractions).

- $f_{LH}(x, y) = \frac{1}{\sqrt{2}} \left[\left(\frac{c+d}{\sqrt{2}} \right) - \left(\frac{a+b}{\sqrt{2}} \right) \right]$. This is the difference of the horizontal rows.

- $f_{HL}(x, y) = \frac{1}{\sqrt{2}} \left[\left(\frac{b-a}{\sqrt{2}} \right) + \left(\frac{d-c}{\sqrt{2}} \right) \right] = \frac{1}{\sqrt{2}} \left[\left(\frac{b+d}{\sqrt{2}} \right) - \left(\frac{a+c}{\sqrt{2}} \right) \right]$. This gives the vertical differences.

- $f_{HH}(x, y) = \frac{1}{\sqrt{2}} \left[\left(\frac{d-c}{\sqrt{2}} \right) - \left(\frac{b-a}{\sqrt{2}} \right) \right] = \frac{1}{\sqrt{2}} \left[\left(\frac{d+a}{\sqrt{2}} \right) - \left(\frac{c+b}{\sqrt{2}} \right) \right]$. This gives the diagonal differences.

Figure 4. 1-level, 2-D inverse discrete wavelet transform (DWT). The up-arrow denotes up-sampling by 2.



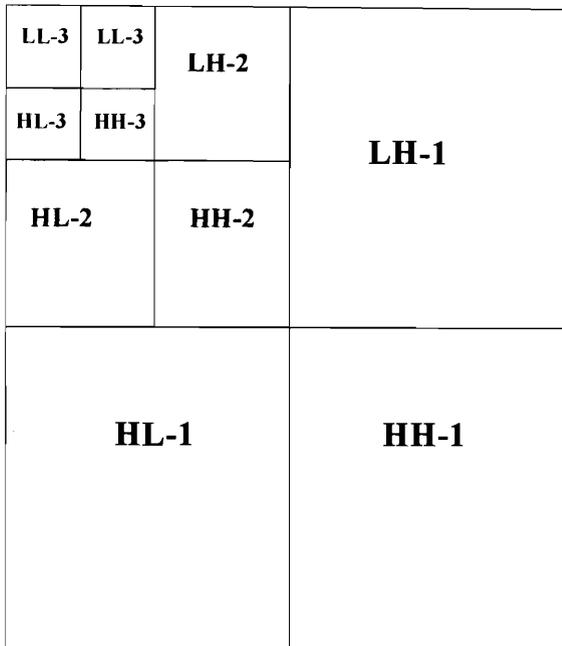


Figure 5. Representation of 3-level, 2-D discrete wavelet transform (DWT). Here LH, HL and HH gives the horizontal, vertical and diagonal differences, respectively, while LL corresponds to average coefficients at each level.

Thus a given image is decomposed into an average (on which DWT could be further applied for higher level of decomposition), horizontal, vertical and diagonal differences. This is represented in *Figure 3*. *Figure 4* illustrates the two dimensional inverse discrete wavelet transform. *Figure 5* pictorially demonstrates the multi-level decompositions of a 2-D data set. Other wavelet transforms will yield weighted averages and differences, as in the one dimensional case. The discerning reader must have noticed words like *down* and *up* sampling, borrowed from the engineering community. The direct product nature of the 2-D wavelet transform mentioned earlier, is quite transparent in *Figures 3* and *4*. We have first applied the filters $h(x)$ and $\tilde{h}(x)$ along the x -axis and subsequently applied the same filters along the y -axis.

Explicit examples of two dimensional wavelet transform and its applications can be found in our upcoming article, *Wavelets: Applications to Image Compression*, in *Resonance*.

Address for Correspondence

Jatan K Modi and
Amit S Phadke
Dharmasinh Desai Institute of
Technology, Nadiad 387 001,
India.

Sachin P Nanavati
National PARAM Super-
computing Facility, Centre for
Development of Advanced
Computing (C-DAC), Pune
University Campus, Ganesh
Khind, Pune 411 007, India.
Email: sachinn@cdacindia.com

Prasanta K Panigrahi
Physical Research Laboratory
Navrangpura
Ahmedabad 380 009, India.
Email: prasanta@prl.ernet.in
URL: www.prl.ernet.in/
~prasanta

