Identities Arising from Finite Differences

One of the basic techniques used in numerical analysis is the analysis of various kinds of finite differences. Let us recall two such differences. If \( f \) is a real-valued function, define the backward difference of \( f \) as

\[
(\nabla f)(x) = f(x) - f(x - 1).
\]

Similarly, define the forward difference of \( f \) as

\[
(\Delta f)(x) = f(x + 1) - f(x).
\]

Note that

\[
(\nabla \nabla f)(x) = f(x) - 2f(x - 1) + f(x - 2)
\]

and

\[
(\Delta \Delta f)(x) = f(x + 2) - 2f(x + 1) + f(x).
\]

One usually denotes these respectively by \( \nabla^2 f \) and \( \Delta^2 f \).

It is easily seen by induction that

\[
(\nabla^n f)(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - i)
\]

and

\[
(\Delta^n f)(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + i).
\]

If the function is sufficiently well behaved, one may use basic theorems of calculus to express the \( n \)-th backward/forward difference of \( f \) at any point as the value of the \( n \)-th derivative of \( f \) at some point. This is much the standard manner in which backward and forward differences are used in numerical analysis. However, there is one further aspect which is perhaps not as well known as it ought to be. Namely, if \( f \) is a polynomial, this is particularly interesting. In fact, we have:
Proposition

Let $f$ be a polynomial of degree $d \geq 0$. Then,

(i) for each $n > d$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - i) = 0 \ \forall \ x$$

and

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x + i) = 0 \ \forall \ x.$$  

(ii)

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} f(x - i) = d!a_d \ \forall \ x$$

and

$$\sum_{i=0}^{d} (-1)^i \binom{d}{i} f(x + i) = d!a_d \ \forall \ x,$$

where $a_d$ is the top coefficient of $f$

Proof.

The mean-value theorem shows that $(\nabla f)(x) = f'(s)$ and $(\Delta f)(x) = f'(t)$ for some $s \in (x - 1, x)$ and $t \in (x, x + 1)$. Since a polynomial is infinitely differentiable, one can apply the mean-value theorem repeatedly to obtain for any arbitrary $n$ that

$$(\nabla^n f)(x) = f^{(n)}(s)$$

and

$$(\Delta^n f)(x) = f^{(n)}(t)$$

for some $s \in (x - n, x)$ and $t \in (x, x + n)$. Therefore, if $n > d$, the corresponding derivatives vanish. This proves (i).

Now, $f^{(d)}$ is a constant since its derivative is zero. This constant is clearly seen to be $d!a_d$ by induction on $d$. Thus, the proposition follows.
The proposition is so general that one has many striking applications. Let us begin with one such.

**Corollary**

(i) If \( n > d \), then

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^d = 0 \quad \forall \ x
\]

and

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x+i)^d = 0 \quad \forall \ x.
\]

(ii)

\[
\sum_{i=0}^{d} (-1)^i \binom{d}{i} (x-i)^d = d! \quad \forall \ x
\]

and

\[
\sum_{i=0}^{d} (-1)^i \binom{d}{i} (x+i)^d = d! \quad \forall \ x.
\]

The corollary is obtained simply by applying the proposition to the polynomial \( f(x) = x^d \). Note that even the special case of \( x = 0 \) gives interesting binomial identities. We go on to give another type of result using the above method.

Note that if the function \( f \) takes integer values at integer points, the \( n \)-th difference also does so for any \( n \). We have:

**Proposition**

*Let \( \theta \) be a positive real number such that \( n^\theta \) is an integer for all sufficiently large natural numbers \( n \). Then \( \theta \) must be a natural number.*

**Proof.**

Consider the function \( f(x) = x^\theta \). As \( f(n) \) is integral for large enough natural numbers \( n \), the same is true of each
Δ^r f Suppose, that θ is not an integer. Write \([θ] = d\) for the largest integer less than or equal to \(θ\). Now

\[(Δ^{d+1} f)(x) = θ(θ - 1) (θ - d)y^{θ-d-1}\]

for some \(y \in (x, x + d + 1)\). This is positive and less than 1 for large positive \(x\). But, it takes integral values for sufficiently large natural numbers \(n\) which is impossible. Hence \(θ\) must be a natural number. This completes the proof.

In conclusion, one can see that the method of finite differences gives interesting results and identities of an algebraic nature also.

---

Information and Announcements

Nobel Prize 2004

**Physics** – “for the discovery of asymptotic freedom in the theory of the strong interaction” to

David J Gross, Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA, USA

H. David Politzer, California Institute of Technology, Pasadena, CA, USA

Frank Wilczek, Massachusetts Institute of Technology (MIT), Cambridge, MA, USA

**Chemistry** – “for the discovery of ubiquitin-mediated protein degradation” to

Aaron Ciechanover, Technion – Israel Institute of Technology, Haifa, Israel

Avram Hershko, Technion – Israel Institute of Technology, Haifa, Israel

Irwin Rose, University of California, Irvine, CA, USA

**Physiology or Medicine** – “for their discoveries of odorant receptors and the organization of the olfactory system” to

Richard Axel, Columbia University, New York, NY, USA; Howard Hughes Medical Institute

Linda B Buck, Fred Hutchinson Cancer Research Center Seattle, WA, USA; Howard Hughes Medical Institute