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Scalars and Vectors in Physics – II

In Part I, the nature of scalars and vectors was discussed for the Newtonian world. However, in the relativistic world, physical quantities follow Lorentz transformations. This change of transformation from Galilean to Lorentz results in interesting distinctions in their properties: mass and time are no longer scalars, time and space become intertwined, etc. This part discusses the nature of scalars and vectors in the relativistic world.

Introduction

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As long as we have two inertial frames that move relative to each other in the Newtonian world, Newtonian laws are valid in both inertial frames and Galilean transformations allow us to transform various physical quantities from one frame to another. This further requires that the laws of physics (within the domain of Newtonian mechanics) be expressed only in terms of scalars and vectors and, in general, tensors. With the advent of the special theory of relativity, it was realized that Newtonian mechanics holds good only for relative velocities that are low compared with the velocity of electromagnetic radiation in vacuum. Since then, the invariance principle has been generalized to include observers moving relative to each other with arbitrary uniform velocities (inertial reference frames). In fact, it is one of the two postulates of the special theory of relativity that physical laws must remain the same for all inertial reference frames. This is known as the principle of relativity. In this sense, Galilean invariance holds good only in the limiting case of relative velocity tending to zero.

Allowing all uniformly moving observers puts restrictions on the scalars and vectors that can be used in the equations and laws of physics. The principle of relativity applied to Maxwell equations leads to the invariance of the speed of electromagnetic radiation in vacuum, and hence forces one to adopt Lorentz transformations in preference to Galilean transformations.

Keywords
Scalars, four-vectors, Lorentz transformation, special relativity.

Let us consider two inertial coordinate systems, say S and S$'$. Let us assume that S$'$ is moving with respect to S with a velocity u in the x direction. According to Galilean transformations, observations on these two frames will be related by:

Galilean invariance holds good only in the limiting case of relative velocity tending to zero.

$$x' = x - ut,$$
$$y' = y,\ z' = z,\ t' = t. \tag{1}$$

Thus, time is invariant in the Newtonian world under Galilean transformation. So are force and acceleration. Although velocity and momentum do not retain the same magnitude, they can be related by Galilean transformation to a different frame of reference.

In contrast, Lorentz transformation gives us the following connections between these two observers.

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}},\ t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}},$$
$$y' = y,\ z' = z. \tag{2a}$$

On defining new coordinates $x_0 = ct$, and $x_1 = x$, $x_2 = y$, $x_3 = z$, and the corresponding primed ones, this can be put in a more compact form as

$$x_0' = \gamma(x_0 - \beta x_1),\ x_1' = \gamma(x_1 - \beta x_0),$$
$$x_2' = x_2,\ x_3' = x_3, \tag{2b}$$

where

$$\beta = u/c,\ \gamma = (1 - u^2/c^2)^{-1/2} = (1 - \beta^2)^{-1/2}. \tag{3}$$

Lorentz transformation can be regarded as a prescription for transforming a four-vector from one inertial frame to another. One can immediately notice that time is not invariant in the relativistic world. Also, the space coordinate x' is connected with the time coordinate t. Similarly, the time coordinate t' is connected with the space coordinate x. Thus, space and time are not independent, as in the Galilean world; they are mixed or intertwined in the relativistic world.

Time is invariant in the Newtonian world under Galilean transformation but time is not invariant in the relativistic world.

Space and time are mixed or intertwined in the relativistic world. Thus, the scalars and vectors of Newtonian mechanics turn out to be inadequate for Lorentzian observers.

This intertwined space and time in the relativistic world has far reaching consequences. Mass, which is a scalar in the Newtonian world and invariant under Galilean transformation, no longer retains its constant value for different inertial observers. Similarly the magnitude of a vector (length) changes from observer to observer. Thus, the scalars and vectors of Newtonian mechanics turn out to be inadequate for Lorentzian observers, and cannot appear as such in relativistic equations.

This mixing also led to the conclusion that Lorentz transformation may be regarded as a rotation in Minkowski space, needing a four-dimensional spacetime coordinate system. In the Minkowski space, three of the dimensions are the usual x, y, and z, and the fourth one is time. Ordered sets of four elements transforming according to vector laws (tensors of rank 1) under Lorentz transformations have been given the name *four-vectors* or *world vectors.*

Invariance

As mentioned in part I, invariance plays an important role in the Newtonian world [1]. All laws of physics remain invariant under the Galilean transformation. A similar situation exists in the relativistic world where all laws of physics remain invariant under Lorentz transformations. This requirement can be fulfilled by only using Lorentz invariant scalars, vectors and tensors in the laws of physics.

Ordered sets of four elements transforming according to vector laws under Lorentz transformations are called *four-vectors* or *world vectors.*

Invariance of physical laws is a critical aspect of physics. Any equation of physics that does not follow this invariance cannot get the status of a law. This requirement leaves us no choice but to use only Lorentz-invariant quantities in physical laws. To give an analogy, scientists would define a person with his/her date of birth, not age, as it remains invariant with respect to time. Similar considerations are chosen in defining physical quantities.

Lorentz Scalars

The concept of a physical scalar, however, remains the same for

Newtonian as well as relativistic worlds. We thus look for scalars which retain their magnitude under relativistic transformations. Only such scalars can occur in the equations and laws of physics which remain invariant for all inertial observers.

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Consider all the Galilean scalars and let us see which of them are *Lorentz scalars* or *world scalars* (the term 'world coordinates' was coined by Minkowski for spacetime coordinates, hence the term world scalars). Since mass, time, and length are not world scalars, several other Galilean scalars such as energy, volume, pressure, etc. are not world scalars either. The speed c of electromagnetic radiation in vacuum is a world scalar, the same for all observers.

Electric charge remains constant regardless of the relative motion, and is invariant under Lorentz transformations, and thus a Lorentz scalar. This is confirmed by experiments. This is the reason that atoms and molecules remain electrically neutral with equal number of electrons and protons. This is in spite of the fact that electrons move quite differently from protons. Also, electrons in different orbitals move with different speeds, in the sense of Bohr orbitals. Thus, Coulomb's law does not depend on the random motion of charges in an object, but only on the total amount of charge present. However, electric charge density is not a world scalar because it involves volume in its definition.

Since charge is a Lorentz scalar, independent of motion, and the electric field is always stationary relative to the observer, there are no relativistic effects due to the interactions between them. Experiments with the cathode ray tube indicate that the force on the electrons remains the same when they move through the electric field, no matter what the speed. All relativistic effects related to charges are due to their interaction with the magnetic field.

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Lorentz Vectors

Galilean vectors are inadequate to serve as vectors under Lorentz transformations for the simple reason that a Galilean vector has

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three components while the latter require four components in four-dimensional spacetime. Thus the least one has to do is to add one more component to a Galilean vector and construct a four-dimensional vector.

However, the mere addition of one more component does not always produce a Lorentz vector. This is because the four components of a *Lorentz vector or world vector* must transform from one inertial frame to another through specified Lorentz transformation equations. It turns that some of the Galilean vectors, such as velocity, linear momentum, electric current density, wave vector, etc., can be made world vectors by the addition of a suitable fourth component. But there are some others, particularly electric field and magnetic field, from which no Lorentz vector can be constructed, though they can be combined to form a tensor of rank two.

We shall follow the standard convention according to which Galilean vectors are denoted by bold face letters such as **u**, **v**, or by their components u_i, v_j, with u, v denoting their classical magnitudes, respectively. Lorentzian four-vectors will be denoted by U^μ, V^ν etc[2]. Latin indices take values 1, 2, 3, while Greek indices take values 0, 1, 2, 3, where 0 denotes the time component and 1, 2, 3 denote the space components. A four-vector is thus denoted by

$$U^\mu = (U^0, U^1, U^2, U^3) = (U^0, \mathbf{U}) \cdot \tag{4}$$

[2] We shall deal only with cartesian four-vectors in flat spacetime and use superscripts for their components for simplicity. Thus we shall not introduce covariant indices and raising and lowering of indices.

The first four-vector that one encounters is the spacetime interval. Let us denote by X^μ, $\mu = 0,1,2,3$, the four-dimensional coordinates chosen by an observer, where $X^0 = ct$ and $(X^1, X^2, X^3) = \mathbf{x}$ is the Galilean position vector. Here c is the speed of electromagnetic radiation in vacuum and t is the classical time coordinate. Let

The first four-vector that one encounters is the spacetime interval.

$$dX^\mu = (dX^0, dX^1, dX^2, dX^3) = (c\,dt, dX^1, dX^2, dX^3) \tag{5}$$

be the components of the interval between two neighbouring spacetime events (X^0, X^1, X^2, X^3) and $(X^0 + dX^0, X^1 + dX^1, X^2 + dX^2, X^3 + dX^3)$ as measured by the observer. Then for all the Galilean observers (stationary relative to each other), dt and $[(dX^1)^2 + (dX^2)^2 + (dX^3)^2]^{1/2}$ are separately invariant.

However, according to special relativity, these two quantities are not separately invariant, but

$$ds^2 = (dX^0)^2 - [(dX^1)^2 + (dX^2)^2 + (dX^3)^2] \qquad (6)$$

is invariant, that is, the same for all inertial observers, invariant under Lorentz transformations.

The 'magnitude' of a four-vector is defined through the scalar product of a four-vector with itself.

Just like $\sqrt{x^2 + y^2 + z^2}$ defines the length of a vector, $\sqrt{c^2t^2 - x^2 - y^2 - z^2}$ defines the 'length' of a four-vector. This length or magnitude remains invariant under Lorentz transformation. This is a condition for all four-vectors and is similar to the three dimensional vector which remains invariant under rotation of the coordinate system.

The *inner product* or *scalar product* of two four-vectors U^μ, V^ν is defined as

$$U^\mu V^\mu = U^0V^0 - [U^1V^1 + U^2V^2 + U^3V^3] = U^0V^0 - \mathbf{u.v}, \qquad (7)$$

where we have used Einstein's summation convention of summing over repeated indices, and **u.v** is the Galilean scalar product in 3-D Newtonian space. The 'magnitude' of a four-vector is therefore defined through the scalar product of a four-vector with itself, and it is the square root of

$$U^\mu U^\mu = U^0 U^0 - [U^1 U^1 + U^2 U^2 + U^3 U^3] = (U^0)^2 - |\mathbf{u}|^2. \qquad (8)$$

Obviously, the magnitude can be real (time-like interval) or imaginary (space-like interval). It can be shown that the magnitude of a four-vector and the scalar product of two four-vectors defined above are invariant under Lorentz transformations.

A few more world vectors, other than the infinitesimal spacetime interval four-vector defined in (5), are mentioned below.

(i) The 'four-velocity' has the components (γc, $\gamma \mathbf{v}$), where $\mathbf{v}$ is the classical 3-D velocity vector and γ is defined in (3). The four-velocity vector is denoted by

$$U^{\mu} = (U^0, \mathbf{U}), \text{ where } U^0 = \gamma c, \mathbf{U} = \gamma \mathbf{v}. \tag{9}$$

The associated invariant quantity is

$$(U^0)^2 - [(U^1)^2 + (U^2)^2 + (U^3)^2] = \gamma^2 c^2 - \gamma^2 v^2 = c^2. \tag{10}$$

(ii) The 'four-momentum', denoted by P^{μ}, has the components

$$P^{\mu} = (P^0, \mathbf{P}) = (E/c, \mathbf{p}), \tag{11}$$

where E is the energy and $\mathbf{p}$ the Newtonian linear momentum vector. The associated invariant magnitude is $(P^0)^2 - p^2$ with $p = |\mathbf{p}|$. For a particle of rest mass m_0, traveling with velocity $\mathbf{v}$, we have the relativistic equations

$$E = \gamma m_0 c^2, \mathbf{p} = \gamma m_0 \mathbf{v}, \tag{12}$$

so that

$$(P^0)^2 - p^2 = m_0^2 c^2, \tag{13}$$

which is indeed a Lorentz scalar.

(iii) The 'four-vector current density' is defined as

$$J = (J^0, \mathbf{J}) = (c\rho, \mathbf{J}), \tag{14}$$

where ρ is the electric charge density and $\mathbf{J}$ the electric current density. To show that $(J^0)^2 - \mathbf{J}^2$ is invariant, we write the transformation of the components, replacing the spacetime coordinates in (2b) by the components J^{μ}, for uniform motion along the X^1-axis, as

$$\begin{aligned} &J^{0\prime} = \gamma(J^0, \beta J^1), \quad J^{1\prime} = \gamma(J^1 - \beta J^0), \\ &J^{2\prime} = J^2, \; J^{3\prime} = J^3. \end{aligned} \tag{15}$$

On working out $(J^{0'})^2 - (J^{1'})^2$ and noting that

$$\gamma^2 (1-\beta)^2 = 1, \tag{16}$$

we see that it reduces to $(J^0)^2 - (J^1)^2$. The point to note here is that charge density is charge per unit volume. For a moving observer, the dimension of volume along the direction of motion will be contracted, enhancing the charge density ρ' by a factor of γ. Further, ρ and **J** mutually get mixed up for a moving observer, each giving rise to the other too. The observer S may have a static charge density and a static current density. For the observer S′, it appears as a moving charge, equivalent to a transient current, and a moving current gives rise to a charge density. It is this which is reflected in the first two equations of (15). Therefore the expression under question remains invariant.

(iv) The 'four-vector potential' is defined as

$$A^\mu = (A^0, \mathbf{A}) = (\phi, \mathbf{A}), \tag{17}$$

where ϕ is the scalar potential and **A** the vector potential of classical electrodynamics.

(v) The 'four-wave vector' is a Lorentz vector whose time component is related to frequency ω and whose space component is the 3-D wave vector **k** of a wave. It is defined as

$$K^\mu = (K^0, \mathbf{K}) = (\omega/c, \mathbf{k}). \tag{18}$$

The associated invariant quantity is $(\omega^2/c^2) - k^2$ or $(\omega^2 - c^2k^2)$, which will be seen to have the same value for all inertial observers. If one observer measures the values ω and k for a wave, the values of frequency and (magnitude of) wave vector measured by another inertial observer would either be both larger or both smaller than ω and k, respectively, so that the quantity $\omega^2 - c^2k^2$ has the same value.

In general, it is seen that each four-vector is made up of one Galilean scalar and one Galilean vector, which respectively form the time and space components of the four-vector. In other

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words, the time component of a four-vector is a Galilean scalar and its space component is a Galilean vector.

Algebra of Four-Vectors

The algebra of four-vectors has the same operations as for the Galilean vectors, except that we are now dealing with ordered sets of four numbers, and that there is no place for concepts like direction and magnitude in the case of four-vectors, or in fact, for all generalized vectors. We may still define the square root of $(A^0)^2 - [(A^1)^2 + (A^2)^2 + (A^3)^2]$ as the *magnitude* of the four-vector A^μ. Then we can see from (10) that *every* four-velocity has the same magnitude c.

Once four-vectors have been defined in the above manner, we can go on to other developments such as the scalar product of two four-vectors. If $A^\mu = (A^0, \mathbf{A})$ and $B^\mu = (B^0, \mathbf{B})$ are two four-vectors, their scalar product is denoted simply by $A.B$ and is defined as

$$A.B = A^0B^0 - [A^1B^1 + A^2B^2 + A^3B^3] = A^0B^0 - \mathbf{A}.\mathbf{B}, \qquad (19)$$

where the last term on the right side is the familiar scalar product of Galilean vectors. Thus the scalar product of the four-wave vector K^μ and the spacetime four-vector X^μ is

$$K.X = K^0X^0 - \mathbf{k}.\mathbf{x} = \omega t - \mathbf{k}.\mathbf{x}. \qquad (20)$$

One can recognize that this is the phase factor in the mathematical representation of a traveling wave having a wave vector $\mathbf{k}$ and angular frequency ω. Thus in this notation, the wave could be represented by $\exp(iK.X)$ which stands for $\exp[i(\omega t - \mathbf{k}.\mathbf{x})]$.

It should be noted that the scalar product of two four-vectors is invariant for all inertial observers, stationary or moving. The proof of this statement is left as an exercise [2]. As remarked in Part 1 [1], it is not possible to define vector product of four-vectors. This means that one cannot construct a four-vector from two given four-vectors.

Examples

In the previous article [1], we mentioned that several mathematical expressions representing physical situations can be derived using a combination of experimental observations and mathematical logic. Let us consider here some examples requiring the use of Lorentz scalars, Lorentz vectors and Lorentz invariance. We shall obtain here the form of the Lagrangian of (a) a freely moving particle, and (b) a charged particle moving in an electromagnetic field.

The Lagrangian of a free particle can depend on its mass and velocity, and must be independent of its spacetime coordinates for a uniformly moving particle.

In the case of a moving particle, the total action A between points is given by the action integral,

$$A=\int_{\tau_1}^{\tau_2} \gamma L \, d\tau, \tag{21}$$

where L is the Lagrangian, τ the proper time with the two particular values τ_1 and τ_2 at two points of the path, and γ is defined in (3). The path of the particle is determined by the condition that the action integral must be an extremum for this path. However, we are not concerned with that aspect here. We wish to obtain the form of the Lagrangian.

The action integral A and the proper time τ are both Lorentz invariants. Thus, from (21), we note that the product γL must also be a world scalar.

(a) Free particle: What are the available parameters on which the Lagrangian can depend? The Lagrangian of a free particle can depend on its mass and velocity, and must be independent of its spacetime coordinates for a uniformly moving particle. The only invariant function that can be constructed from its velocity U^μ is $U.U=c^2$ (see (10)). The mass of a particle is not invariant, though its rest mass m_0 is. Since a Lagrangian has the dimensions of energy, we expect it to be a linear function of its rest mass and a quadratic function of the velocity. We thus get the unique form of the Lagrangian, apart from constant factors, from these invariance arguments to be

$$\gamma L_f = k m_0 U.U \quad \text{or} \quad L_f = k m_0 c^2 (1 - v^2/c^2)^{1/2}, \tag{22}$$

where k is a constant and L_f stands for the Lagrangian of a free particle. Conventionally, the constant is taken to be $k = -1$.

(b) Particle in an electromagnetic field: Electromagnetic fields **E** and **B** can be expressed in terms of the scalar and vector potentials ϕ and **A**. We have mentioned in (17) that (ϕ, **A**) is a Lorentz invariant four-vector, which can be denoted by A^μ. The presence of an electromagnetic (em) field will merely cause a term L_{int} due to the interaction of the electromagnetic field to be added to the free particle Lagrangian of (22), so that the Lagrangian L_{em} of an otherwise free particle in the em field can be written as

$$L_{\text{em}} = L_f + L_{\text{int}}. \tag{23}$$

This is because as the em field is made weaker and weaker, the full Lagrangian L must reduce to L_f. Thus our task is to obtain the form of L_{int}.

We would need another limiting case, that of a low velocity charged particle in the presence of an em field. As the velocity of charge q tends to zero, the interaction with the magnetic field (that is, with the vector potential **A**) reduces, and the interaction is dominated by the electric field (or the scalar potential ϕ). Thus in the limiting case of $\mathbf{v} \to 0$, L_{int} must tend to $-q\phi$. The negative sign occurs here due to the fact that the Lagrangian is $T-V$, where T is the kinetic energy and $V = q\phi$ is the potential energy.

We also note that L_{int} can depend on the charge q, velocity four-vector U^μ and the four-vector potential A^μ. The only Lorentz invariant quantity that can be formed from these is $qU.A$. Thus we have

$$\begin{aligned} \gamma L_{\text{int}} &= kq\, U.A \\ &= kq\,(U^0 A^0 - \mathbf{u}.\mathbf{A}) \\ &= kq\,(\gamma c\phi - \gamma \mathbf{v}.\mathbf{A}), \end{aligned} \tag{24}$$

where k is a constant. This gives

$$L_{\text{int}} = kq\,(c\phi - \mathbf{v}.\mathbf{A}). \tag{25}$$

In the limit as $\mathbf{v} \to \mathbf{0}$ this must reduce to $-q\phi$, which is possible if $k = -1/c$. This finally gives

$$L_{\text{int}} = -q\phi + (q/c)\,\mathbf{v}.\mathbf{A}. \tag{26}$$

The full Lagrangian thus becomes [3]

$$L = -m_0c^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - q\phi + \frac{q}{c}\mathbf{v}.\mathbf{A}. \tag{27}$$

(c) **Dirac equation for an electron:** The Schrödinger equation is invariant under Galilean transformations, but not under Lorentz transformations. Thus the results obtained from the Schrödinger equation would not hold good in different inertial frames. The special theory of relativity departs from classical mechanics in the domain of moving inertial observers. Dirac combined these two aspects and obtained an equation which would describe the behaviour of a microscopic particle such as an electron and which, at the same time, would be invariant under Lorentz transformations. He used exactly the logic described above and required that the Hamiltonian of an electron, which represents energy, be symmetric in all the four spacetime coordinates. Schrödinger's time-dependent equation involves second derivatives in space coordinates but a first derivative in the time coordinate. Dirac wrote the simplest possible Hamiltonian involving space and time coordinates in a symmetric manner, and he could achieve this with first derivatives in all the four coordinates[4]. One of the consequences, as is well-known, was the fact that electron spin arose as a natural consequence of Dirac's relativistic equation, but we will not go into any more details here. We may remark, however, that because of this difference (between Galilean invariance and Lorentz invariance), the Dirac equation is not only more relevant but aesthetically and conceptually much more beautiful and satisfying than the Schrödinger equation.

Suggested Reading

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