Logarithmic Spiral – A Splendid Curve

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Due to its various peculiarities, logarithmic spiral drew the attention of mathematicians and was called 'spiral mirabilis' by Jacob Bernoulli I. In this article some properties of logarithmic spiral have been described along with the appearance as well as applications of the curve in art and nature.

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After the discovery of analytical geometry by Rene Descartes (1596-1650) in 1637, the custom of representing various curves with the help of equations came into vogue. Logarithmic spiral was one of those curves which at that time drew the attention of mathematicians. During that time the polar equation of logarithmic spiral was written as $\ln r = \theta$ where 'ln' stands for natural logarithm, i.e. logarithm with base e (then called as hyperbolic logarithm). At present that equation is written in the form $r = e^{a\theta}$, where θ is measured in radians (one radian is approximately equal to 57 degrees). Logarithmic spiral looks like *Figure* 1.

The constant 'a' represents the rate of increase of the spiral. When $a > 0$, then r increases in the anti-clockwise sense and left-handed spiral is generated (*Figure* 1).

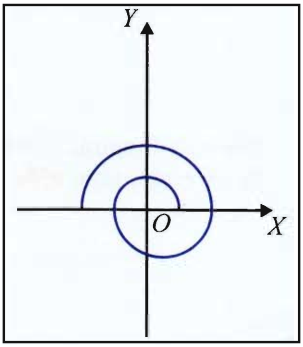


Figure 1. Logarithmic spiral.

On the other hand, if $a < 0$ then r decreases in the anti-clockwise direction and we get a right-handed spiral (*Figure* 2). We know that the derivative of e^x is also e^x. Various properties of logarithmic spiral depend on this property of e^x.

Properties of Logarithmic Spiral

1. The most important property of a logarithmic spiral is that r (i.e. the distance from the origin) increases

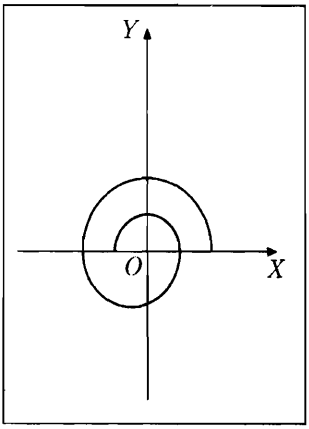


Figure 2. Right-handed spiral

proportionately i.e. remain in G.P. with increase of θ. The reason for this is $e^{a\phi}$ acts as common ratio in the relation $e^{a(\theta+\phi)} = e^{a\theta}\ e^{a\phi}$.

2. If from any point on a logarithmic spiral one starts spiralling inwards along the curve then an infinite number of complete rotations are required to reach the origin, but the distance traversed will be finite (see *Box* 1). This property was discovered by Evangelista Toricelli (1608-1647), pupil of Galileo Galilei (1564-1642). He showed that the distance (along the curve) of any point F from the origin is equal to the distance of the point of intersection of the tangent at P with the y-axis

Box 1

We know that if θ_1 and θ_2 are the values of θ for two points on a curve whose equation is of the form $r = f(\theta)$ and s be the length of the arc between those two points, then,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

For logarithmic spiral, $r = e^{a\theta}$. Therefore

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = a e^{a\theta}.$$

Therefore for logarithmic spiral,

$$s = \int_{\theta_1}^{\theta_2} \sqrt{(e^{a\theta})^2 + (a e^{a\theta})^2} = \int_{\theta_1}^{\theta_2} \left(\sqrt{1+a^2}\right) e^{a\theta}\mathrm{d}\theta$$

$$= \frac{\sqrt{1+a^2}}{a}\left(e^{a\theta_2} - e^{a\theta_1}\right)$$

For $a > 0$ spiral is left-handed. Now, if we decrease θ_1 infinitely keeping θ_2 fixed (i.e. $\theta_1 \to -\infty$), then $e^a\theta_1 \to 0$. In that case,

$$s = \frac{\sqrt{1+a^2}}{a} e^{a\theta_2} = \frac{\sqrt{1+a^2}}{a} r_2 \qquad (1)$$

Since both a and r_2 are finite, then s is always finite. i.e. to reach the origin from any point on a logarithmic spiral one has to travel a finite distance.

from the origin (*Figure* 3). This discovery of Toricelli was the first instance of rectification, i.e. the calculation of the length of an arc of a non-algebraic curve.

3. Any line segment drawn through the origin always intersects a logarithmic spiral at equal angles (*Figure* 4). For this reason a logarithmic spiral is also known as an equiangular spiral. If we put $a = 0$ in the equation of an equiangular spiral, then we get $r = 1$ which is the equation of a unit circle. So, circle is a special type of equiangular spiral whose rate of growth is zero.

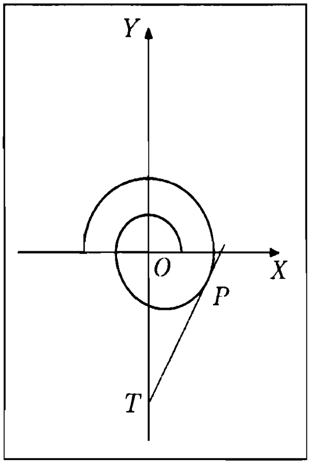


Figure 3. Distance of P from O is equal to PT.

4. An interesting property of a logarithmic spiral is that it remains unaltered under many geometrical transformation. The transformation by which a point (r, θ) is mapped to a point $(1/r, \theta)$ is known as inversion. Usually a curve changes to a large extent by inversion. But, the inversion of a logarithmic spiral gives us another logarithmic spiral which is the mirror image of the first one, i.e. a left-handed spiral is converted into a right-handed one and vice-versa.

5. The rate of change of direction of the curve at any point on it is called the curvature at that point. The reciprocal of curvature is known as radius of curvature.

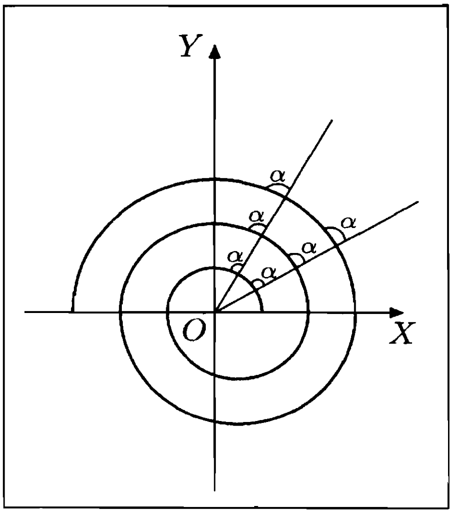


Figure 4. Equiangular spiral.

Jacob Bernoulli I wished that a logarithmic spiral be engraved on his tombstone with the sentence 'Eadem mutata resurgo' (Though changed, I shall arise the same).

If we draw a perpendicular to a tangent of the curve at the point of contact (in the concave side of the curve) and move along the perpendicular (i.e. normal) through a distance equal to the radius of curvature then we reach a point known as centre of curvature. The locus of centre of curvatures of different points on a curve is called the evolute of that curve. It is interesting to note that the evolute of a logarithmic spiral is itself.

6. The locus of the foot of perpendiculars of the orthogonal projections of the tangents of a curve drawn from the pole is known as the pedal of that curve. The pedal of a logarithmic spiral is the logarithmic spiral itself.

7. The envelope formed by the reflections by the curve of the rays drawn from the pole is called the 'caustic' of the curve. The caustic of a logarithmic spiral is the spiral itself.

The above mentioned properties of a logarithmic spiral surprised Jacob Bernoulli I (1654-1705), a mathematician member of the famous Bernoulli family, so much that he christened logarithmic spiral as 'spira mirabilis' (The marvellous spiral) and wished that a logarithmic spiral be engraved on his tombstone with the sentence 'Eadem mutata resurgo' (Though changed, I shall arise the same) written below it. This desire of Jacob was like the desire of Archimedes (see *Box* 2). Anyway, Jacob's desire was fulfilled almost exactly. But, the artisan of his tombstone, either out of ignorance or to make his work easier, engraved an Archimedian spiral (see *Box* 3) and not a logarithmic spiral on it! Tourists at Munster Cathedral of Basel (Switzerland) even today can see the Archimedian spiral and the inscription under it on the tombstone of Jacob Bernoulli I.

Logarithmic Spiral in Nature

Apart from logarithmic spiral no other curve seems to have attracted the attention of scientists, naturalists and

Box 2. Archimedes' Wish

If a plane figure like *Figure* A is rotated about AD where AD=BD=DC, then we get a right circular cone inscribed in a hemisphere which is again inscribed in a right circular cylinder. Archimedes proved that the ratio of the volumes of the cone, the hemisphere and the cylinder is 1:2:3. This discovery made Archimedes so happy that he wished that a right circular cone inscribed in a hemisphere which is further inscribed in a right circular cylinder be engraved on his tombstone. His desire was fulfilled. For this reason, the above-mentioned theorem (the ratio of the volumes of three solids is 1:2:3) is known as Tombstone Theorem.

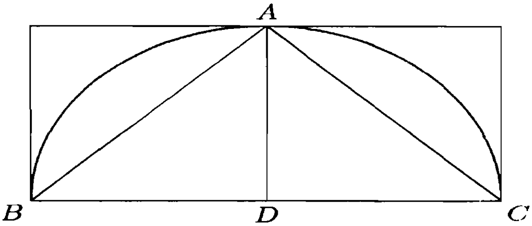


Figure A.

artists alike. British naturalist D A W Thomson (1860-1948) has made detailed discussions of the natural tendency of growth (in the form of logarithmic spiral) of conch, horn, tusk etc. in his book *On Growth and Form.* It should be mentioned here that the first property of logarithmic spiral (already mentioned) is found to be true in the structure of Nautilus shell.

After the resurgence of the ideas about the relation between Greek art and mathematics in the initial years of the twentieth century, new light was thrown on the logarithmic spiral. In 1914, Sir Theodore Andrea Cook, in his book *The Curves of Life*, discussed the role of various spiral curves in the field of art as well as in nature.

Again, the book *Dynamic Symmetry* of Jay Hambidge (published in 1926) has also influenced several artists for a long time. Hambidge has written the book keeping 'golden section' or 'golden ratio' (see *Box* 4) in his mind. Rectangles whose lengths and breadths are in the ratio ϕ:1 (where ϕ is the golden number) are known as 'golden rectangles'. According to many artists, dimensions (length and width) of golden rectangles are most

Box 3. Archimedian Spiral

The equation of an Archimedian spiral is $r = a\theta$. Perhaps Archi-medes discovered various properties of this curve and hence the curve bears his name.

Box 4. Golden Section

If a line segment is divided into two parts such that the ratio of the entire segment to the larger part is equal to the ratio of the larger part to the smaller part, then that division is known as 'golden section'. For instance, in *Figure* B a segment AB of unit length is divided by a point C in such a way that the length of the greater part AC is x and that of the smaller part is $(1-x)$. Therefore

$$AC : CB = x : (1-x).$$

Again, $AB : AC = 1 : x$. If, C divides AB in golden section then

$$1/x = x/1-x.$$

Therefore

$$x^2 + x - 1 = 0.$$

The postive root of the above quadratic equation is $\frac{\sqrt{5}-1}{2}$, i.e. approximately 0.61803. Golden number is the receiprocal of this positive root, i.e. nearly 1.61803, Golden number is denoted by ϕ. Therefore

$$\phi = 1.61803(\text{nearly}).$$

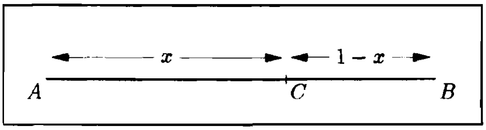
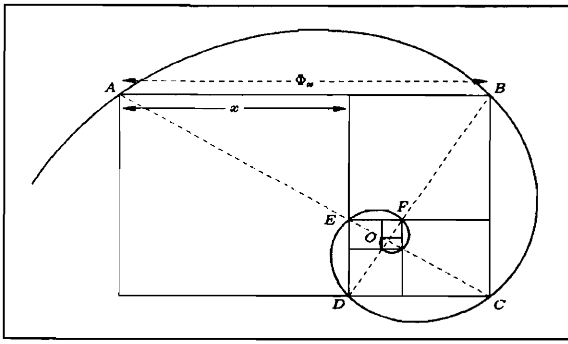


Figure B.

beautiful from the aesthetic point of view. For this reason, golden section has played a major role in architecture. It is interesting to note that if the breadth of a golden rectangle is taken as the length of another rectangle then that rectangle will also be a golden rectangle. For instance, if a square of side x is cut off (*Figure* 5) from a golden rectangle of sides ϕx and x (i.e. the sides are in the ratio $\phi : 1$) then the new rectangle formed by this process will also be a golden rectangle because the length and breadth of the new rectangle will be x and $\phi x - x$ respectively and

$$\phi x - x = (\phi - 1)x = \frac{x}{\phi} \quad \left(\text{since} \quad \phi - 1 = \frac{1}{\phi}\right).$$



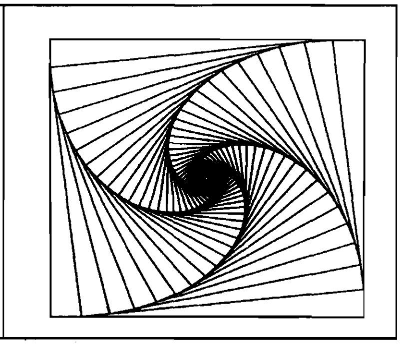


Figure 5 (left).
Figure 6 (right).

Ratio of the length and breadth of the new rectangle $= x : x/\phi = \phi : 1$. In *Figure* 5, A,B,C,D,E and F lie on a logarithmic spiral. In this process, starting with a golden rectangle we can obtain an infinite number of smaller and smaller golden rectangles whose areas gradually tend to zero.

Now, let us conclude our discussion with a beautiful problem. Suppose, four bugs are placed at the four corners of a square. As soon as the bugs receive a signal, each of them starts moving in such a manner that at any instant the motion of each bug is directed towards its neighbouring bug. Then it will be found that the path traced out by each bug is a logarithmic spiral and those spirals converge to a point at the centre (point of intersection of the diagonals) of the square (*Figure* 6). This is the famous 'Four Bug Problem'. Based on this picture many designs have been made.

Suggested Reading

[1] Eli Maor, *e, The Story of a Number*, Universities Press, 1999.

[2] Stuart Hollingdale, *Makers of Mathematics*, Penguin Books, 1989.

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