In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Scalars and Vectors in Physics – I

Scalar and vector quantities are ubiquitous in physics. However, most physics texts at the undergraduate level provide only a brief description of their nature. This creates confusion for many: all magnitudes are scalars and any physical quantity with magnitude and direction is defined as vector. The true test of a scalar or vector quantity comes by testing its nature under Galilean transformations, directed line segment, parallelogram or triangular law of addition. This article covers the nature of scalars and vectors that is appropriate for the undergraduate level. Lorentz scalars and vectors in four-dimensional space will be discussed in the next part.

Introduction

What are the common properties between distance, mass, density, speed, pressure, energy, electric potential, and power? Some readers may be baffled by this question as these quantities are varied in their properties. Thanks to the wonders of physics, we now know that all quantities from this list obey exactly the same set of laws. Yes! They have a group of properties that are common among them and they come under the technical term *scalar*. Similarly, displacement, velocity, force, momentum,

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electric field, and magnetic field have a common set of properties and hence they are called *vectors* [1-5].

Generally a scalar is defined as a physical quantity having only a magnitude, whereas a vector is said to be a physical quantity having a magnitude and a direction. For example, mass of an object does not change, no matter which coordinate system you choose to measure it. In contrast, some other physical quantities must maintain their direction to remain the same. Any change in direction does change the measured numbers. The force acting on an object is an example of such physical quantities where direction is important.

The vector nature of certain physical quantities was known to the earlier natural philosophers such as Galileo and Newton.

The vector nature of certain physical quantities was known to the earlier natural philosophers such as Galileo and Newton. For example, Galileo explained the projectile motion by separating the x- and y-motions. The algebra of vectors is more recent and was mostly developed by W R Hamilton, G G Grassmann, O Heaviside and J W Gibbs in the nineteenth century.

A vector is represented by a directed line segment. The length of the line is directly proportional to the magnitude and the direction of the line represents the direction of the physical quantity. A vector is shown with a tail and a tip and the direction is defined at the tip. The Cartesian coordinate system as well as polar coordinate system can be used to represent a vector. In three dimensions, a vector can be written as a linear combination of **i**, **j**, **k**, the unit orthogonal triad. For example,

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}, \qquad (1)$$

where u_x, u_y, and u_z are the projections of **u** on any Cartesian x, y, and z axes (*Figure* 1). If we apply the inversion operator on the coordinate system, the new coordinates become

$$x' = -x,\ y' = -y, \text{ and } z' = -z. \qquad (2)$$

Thus, a vector **u**, under inversion, becomes vector –**u**. However, under this inversion, vector **u**×**v** does not become –**u**×**v**. This happens as u_x, u_y, and u_z are changed into $-u_x$, $-u_y$, and $-u_z$.

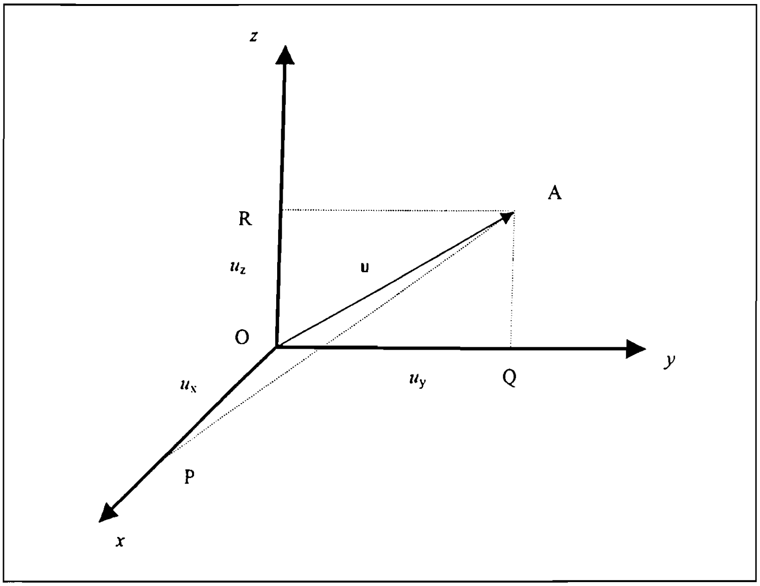
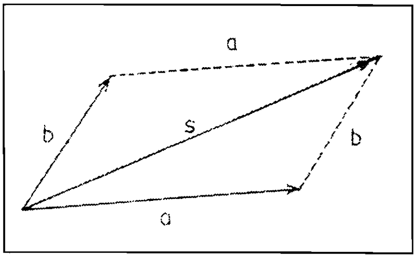


Figure 1. Components of a vector in a Cartesian coordinate system x, y, z.

Similarly, for vector **v**, v_x, v_y, and v_z will be replaced by $-v_x$, $-v_y$, and $-v_z$. The end result of $-\mathbf{u}\times(-\mathbf{v})$ remains the same. Therefore, **u**×**v** is not a true vector; it is called a 'pseudovector'. For this reason, many vectors associated with rotations are labeled as pseudovectors. Thus, torque is actually a pseudovector. This means, it does not reverse itself under mirror image. For our purpose, let us ignore this distinction of the mirror image and consider vectors as well as pseudovectors under the common umbrella of vectors.

Vectors can be displaced by keeping their magnitude and direction intact. It does not change the vector. Vectors follow parallelogram (or triangle) law of addition. This can be defined in simple terms as follows for two vectors **a** and **b**. From the tip of vector a, construct a vector parallel to **b**. The vector from the tail of **a** (or origin) to the tip of the new parallel vector **b** defines the vector sum **s** = **a** + **b** (*Figure* 2). They also follow the commutative law. Thus, **a** + **b** = **b** + **a**.

Figure 2. Vectors a and b their sums.



This brief introduction quickly tells us that there is more to scalars and vectors than defined in most physics textbooks at the undergraduate level. We have divided the article into two parts: the first part deals with non-relativistic mechanics while the second part deals with relativistic mechanics.

The Concept of Invariance

Imagine a physics experiment being performed in a laboratory. Suppose ten scientists, who are stationary or moving with uniform velocities relative to each other [1], that is, in inertial frames of reference, are measuring some scalar quantities concerning this experiment, such as length of a rod, mass of a particle, time elapsed between two events, magnitude of the electric field, etc. Each scientist has the necessary sophisticated equipment with them. Each one has chosen his/her own coordinate systems (Cartesian or general). Would the measurements of the aforesaid quantities by the different scientists be different? In other words, would the measurements depend on the coordinate system chosen?

Now we know from special theory of relativity that, what is described here is not strictly true for moving observers. We are restricting to pre-relativistic mechanics in this Part I, which still holds good for low velocities.

Experience has shown that the measurements should not differ (by more than the limits of experimental error). At the same time, due to the moving frame of reference, velocity and momentum will differ. Although the measurements will differ and even the phenomenon will appear to be different, they should be able to correlate the observations using Galilean transformation (law of velocity addition). The laws of physics will remain valid for all observers. For example, a person (let us call her observer A) sitting in a moving train drops a ball. She can predict the fall using one-dimensional motion. Similarly, another person (let us call her observer B) standing outside and looking at the event will observe a projectile motion for the ball. She should be able to predict this two dimensional motion with the laws of physics. Two different observations for different observers and yet the laws of physics can explain the situation.

Scientists grasped this concept about four centuries ago and empirically formulated a principle of invariance, which is today known as the *principle of Galilean invariance*: *Laws of Physics*

Principle of Galilean invariance: Laws of Physics should remain the same for all inertial observers.

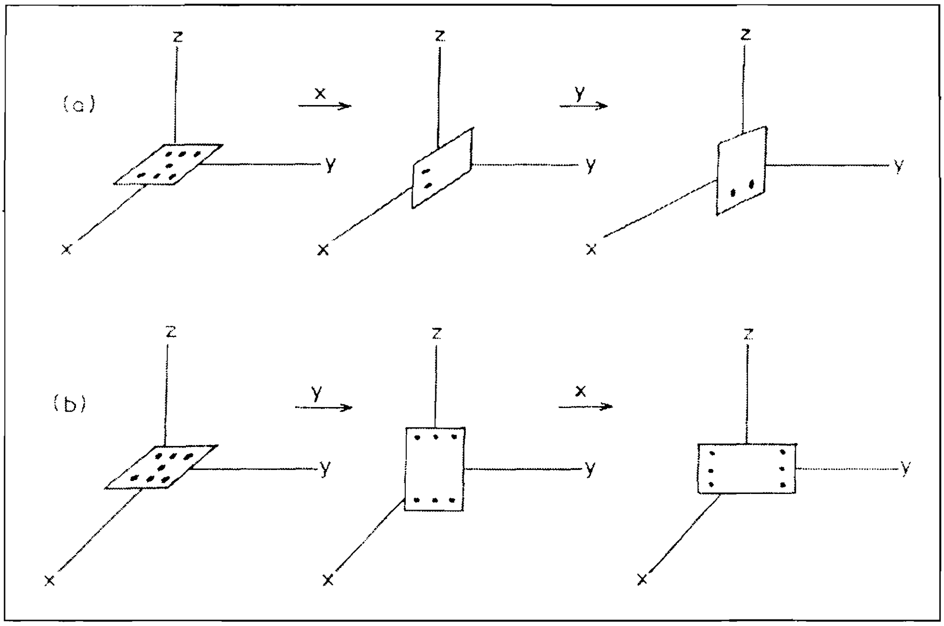
should remain the same for all inertial observers. It is very helpful in studying and communicating physics. All observers irrespective of location can verify a law of physics. It allows physicists to communicate with each other.

Conversely, in the mathematical formulation of physical laws, only those quantities, which either remain invariant under transformation or can be related to each other by Galilean transformation with respect to the change of coordinate system, can occur. These are scalars, vectors, and in general tensors.

Rotation of a Body – A Misconception

Angular velocity is considered a vector quantity. One would be tempted to say that angle of rotation is also a vector. However, this is not the case. Consider the rotation of a body about some axis. Let us define a quantity whose direction is parallel to the axis of rotation, and whose magnitude is equal to the angle of rotation. So if direction and magnitude were the only criteria, this quantity would qualify as a vector quantity. However, two or more such quantities do not add according to the parallelogram law of vectors. To see this, take a book and decide your Cartesian axes, with the origin at the centre of the book and axes parallel to the edges of the book. Rotate the book about the x-axis through $\pi/2$, and then about the y-axis through the same angle $\pi/2$. Note the orientation of the book. Now bring back the book to its original orientation, and perform the same two operations in the reverse order, and you will find that the orientation of the book is different, showing that these operations do not commute with each other. In *Figure* 3, the origin is taken at the centre of the book which lies in the x-y plane, x-axis coming down the book, and y-axis going to the right. The observer is looking toward the book from the (1, 1, 1) direction. Also if you consider any other magnitude, other than π, for the two rotations, treat them as vectors for a moment and obtain their resultant, the result of the actual rotation is not the same as rotating the body about a direction parallel to the resultant vector. Thus finite rotations, in spite of having magnitude and direction, do not follow the

Finite rotations do not follow the commutative law of addition and are not vectors.



commutative law of addition (or parallelogram law), and are not vectors.

Figure 3. Performing two successive rotations on a book about different axes through $\pi/2$ in different order does not result in the same state. (a) Rotation about x-axis followed by that around y-axis; (b) Rotation about y-axis followed by that around x-axis. The rotation follows the right hand screw rule, $x \Rightarrow y \Rightarrow z \Rightarrow x$. Two dots indicate the rear side, lower edge, of the book. The axes remain unchanged, only the book rotates.

Magnitude – Not Always Scalar

As mentioned, a scalar can be represented only by a magnitude, and a vector can be represented by a magnitude and a direction. But the converse is not true. Thus, not everything represented by a magnitude is a scalar, nor a quantity represented by a magnitude and direction always a vector.

Let **u** be a vector, with components u_1, u_2, u_3. These components are magnitudes. Their sum $u_1 + u_2 + u_3$ will also be a number. But that is not enough to qualify it as a scalar, because this sum would depend on the coordinate system of the observer and would not be invariant under coordinate transformations. Different observers may have different coordinate systems, and therefore different values of the components, and thus the sum indicated above would vary from observer to observer. This can be checked easily by considering a two-dimensional vector for simplicity, choosing two different Cartesian coordinate systems, and finding out the sum of the components in both systems.

We need a transformation which will correlate the observations of one scientist with another. This is where Galilean transformation comes handy.

Similarly, an arbitrary quadratic combination of the type $u_1^2 + 2u_2^2 - 5u_3^2$ would not be a scalar as this number would have different values for different observers. However, there is one particular quadratic combination of these components, that is $u_1^2 + u_2^2 + u_3^2$ in the Cartesian system, which is invariant under such Galilean transformations, and hence is a physical scalar. It is the square of the magnitude of the vector **u**.

Vectors

Now consider a vector quantity, such as force, velocity, electric field or magnetic field. These are directed quantities, and hence the observer has to choose a coordinate system. The result of an observer's measurement could be something like this: 'The magnitude of this quantity is xxx, and its direction makes angles α, β, γ with my x-, y-, and z-axes, where my x-axis points towards .... There may be ten scientists observing and measuring the same quantity, each with their coordinate system. The angles α, β, γ measured by different observers would be different since they would depend on their coordinate systems. In other words, the projections of the vector on the three coordinate axes of each observer *would be* different. We need a transformation which will correlate the observations of one scientist with another. This is where Galilean transformation comes handy. All measurements will be related to each other through Galilean transformation. Thus, for a vector, we require some algorithm to arrive at agreement between the measurements of various observers. Moreover, magnitude and direction can only be specified when we talk of real three-dimensional vectors, whereas the definition in terms of transformations is very general and can be used in any arbitrary n-dimensional inner product space.

Though we shall confine ourselves to Galilean and Newtonian mechanics in the familiar three-dimensional space, we make a brief mention of four-dimensional mechanics in special theory of relativity in order to emphasize the importance of correct transformation properties, as against direction and magnitude. When we go to special theory of relativity and four-dimensional spacetime, it is well-known that the mass of a body may have different values depending on the observer. Thus mass is not a Lorentz-invariant quantity. Even if it can be denoted by a number in each observer's frame of reference, it is *not* a scalar in four-dimensional mechanics of special theory of relativity. On the other hand, it is found that charge is a Lorentz-invariant quantity, the same for all observers in uniform relative motion with respect to each other. Hence electric charge is a scalar in four-dimensional mechanics. We shall deal with scalars and vectors in special relativity in the second part of this article.

The set of infinitesimal rotations can be treated as a set of vectors, but the set of finite rotations cannot.

Addition of Vectors

Consider two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, with Cartesian components as shown. The sum of these vectors is defined as a vector $\mathbf{w}$ having components

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3). \qquad (3)$$

It can be shown that if the components of $\mathbf{u}$ and $\mathbf{v}$ transform according to vector transformation laws, then those of $\mathbf{u} + \mathbf{v}$ also do. Note that only in a real three-dimensional space, it is possible to make geometrical representations and diagrams. In such a case, the above rule of vector addition is commonly known as the *parallelogram law of addition*. In the general situation of n-dimensional vectors, whose components may be real or complex, one cannot visualize or make any diagrams. It must also be noted that the vector addition rule of (3) is a basic requirement for $\mathbf{u}$ and $\mathbf{v}$ to be vectors.

This explains why the set of infinitesimal rotations can be treated as a set of vectors, but the set of finite rotations cannot. Here we are constructing two sets of entities, one of all finite rotations and one of all infinitesimal rotations. For each set, an element is a rotation which has the direction of the axis of rotation, taking the positive direction, as per the convention, according to the right hand screw rule, and magnitude equal to the angle of rotation. Thus two infinitesimal rotations $\delta\theta\,\mathbf{n}$ and $\delta\phi\,\mathbf{m}$, where $\mathbf{n}$ and $\mathbf{m}$ are two unit vectors, can be added exactly

Can we construct Galilean scalars and vectors out of two vectors **u** and **v**?

like we do in higher secondary school by the parallelogram law of addition. Now consider two finite rotations $\theta\mathbf{n}$ and $\phi\mathbf{m}$ on a body about the same two directions. There is no other reason why these cannot be treated as vectors except that if we were to treat these as vectors, and obtain their resultant by the parallelogram law of addition, we will get some magnitude and direction for the resultant. But we shall see that the resulting orientation of the body does not agree with this. This implies that finite rotations do not follow the parallelogram law of addition of vectors. This is what we saw in *Figure* 3.

The same can be said about an infinitesimal element of area, which can be treated as a vector, with magnitude equal to the area and direction normal to the area. Here again, the essential concepts of invariance with respect to different observers and vector laws of addition hold good.

Vector Algebra

Can we construct Galilean scalars and vectors out of **u** and **v** defined above? If we construct a combination such as $au_1v_1+bu_2v_2+cu_3v_3$, with arbitrary constants a, b, c, it can be seen by using coordinate transformations that this combination does not retain its form for different coordinate systems, except when $a = b = c$. Taking $a = b = c = 1$, without loss of generality, we define and denote the combination

$$\mathbf{u.v} = u_1v_1 + u_2v_2 + u_3v_3 \qquad (4)$$

as the *scalar product* of two vectors.

We may similarly ask: Given two vectors **u** and **v** as above, can we construct a vector by combining their components suitably? In other words, can we construct three combinations from the components of **u** and **v** which will transform under coordinate changes in exactly the same way as the components of **u** and **v** do? It turns out that we can, and the desired combinations are $u_2v_3 - u_3v_2$, and two others obtained by cyclic permutation of 1, 2, and 3. Now we need a symbol and a name for this contraption!

We call it the *vector product* of two vectors, or popularly the cross product of two vectors, and denote it by $\mathbf{u}\times\mathbf{v}$. Thus,

$$\mathbf{u}\times\mathbf{v} = (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}) \times (v_x i + v_y\mathbf{j} + v_z\mathbf{k}) = (u_y v_z - u_x v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k}. \quad (5)$$

It must be noted that, while the scalar product is commutative, the vector product is not. Thus,

$$\mathbf{u}\times\mathbf{v} = -\,\mathbf{v}\times\mathbf{u}. \quad (6)$$

Geometrically, the vector product is the area of the parallelogram having vectors **u** and **v** for its adjacent sides, and direction normal to the plane containing the two vectors. It must be noted that this definition of the cross product holds good only in a real three-dimensional space, and in some other situations (7-D space) in an abstract manner. On the other hand, the scalar product defined in (4) can be generalized to n-dimensional spaces.

Geometrically, the vector product is the area of the parallelogram having vectors **u** and **v** for its adjacent sides, and direction normal to the plane containing the two vectors.

Scalar and Vector Operators

Scalar and vector operators are introduced at the undergraduate level, and vector calculus is developed, without emphasizing the import of why a particular definition is adopted for these operators. Thus we introduce gradient, divergence, and curl as the three operators in vector calculus. They can act on scalar or vector functions, as the case may be, and may result in either type of function, again depending on the operator. The type of function on which they can act, and the nature of the resulting function, are summarized in *Table* 1.

If $f(\mathbf{r})$ is a scalar function of the position vector (or any other vector variable) $\mathbf{r} = (x_1, x_2, x_3)$, then the action of the gradient operator on it is defined as

$$\nabla f(\mathbf{r}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right) \quad (7)$$

Table 1. Operators of vector calculus, and their operation on functions.

Operators	Type of field on which operator acts	Type of the resulting field
Gradient	Scalar field	Vector field
Divergence	Vector field	Scalar field
Curl	Vector field	Vector field

If we use the differential operator in three-dimensional space, then we can obtain *only one scalar function* and *only one vector function*, called the *divergence* and the *curl* of the function.

One may ask why we cannot define the gradient operator as having components $a\frac{\partial f}{\partial x_1}$, $b\frac{\partial f}{\partial x_2}$, $c\frac{\partial f}{\partial x_3}$, with arbitrary values of constants a, b, c. When we perform a transformation on these three 'components', or expressions, to another coordinate system x_1', x_2', x_3', it is seen that the new 'components' will retain the same form *if and only if* $a = b = c$. This means that in the new coordinate system, the action of the gradient operator can be denoted, taking $a = 1$, by

$$\nabla' f(\mathbf{r}') = \left(\frac{\partial f}{\partial x_1'}, \frac{\partial f}{\partial x_2'}, \frac{\partial f}{\partial x_3'} \right), \tag{8}$$

where ∇' denotes differentiation with respect to components of $\mathbf{r}'$.

Similarly suppose we have a vector field $\mathbf{u}(\mathbf{r})$. Which scalars or vectors can we construct from it? One is of course the scalar product of the vector field with itself,

$$|\mathbf{u}(\mathbf{r})|^2 = \mathbf{u}(\mathbf{r}).\mathbf{u}(\mathbf{r}). \tag{9}$$

If we use the differential operator in three-dimensional space, then we can obtain *only one scalar function* and *only one vector function*, called the *divergence* and the *curl* of the function, and defined and denoted respectively by

$$\nabla.\mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \tag{10a}$$

$$\nabla \times \mathbf{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \tag{10b}$$

Here again, it must be noted that while the gradient and divergence operators can be easily generalized to n-dimensional spaces, the form of the curl operator in (10b) is typical of a real three-dimensional space.

A Simple Transformation

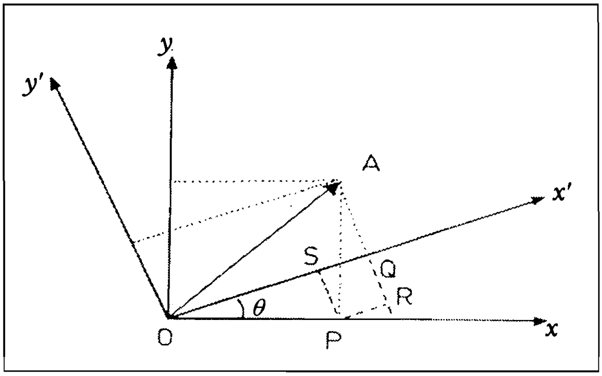
In order to understand the concept of coordinate transformations, we illustrate here a very simple transformation, the rotation of rectangular Cartesian coordinate axes in a two-dimensional plane through a certain angle. Consider a vector such as the velocity of a moving body, or some force, or electric field, and let us restrict ourselves to the two-dimensional plane for the sake of simplicity. In order to measure and specify it, we (observer S) choose a Cartesian coordinate system with axes x and y. We measure the components of the vector along our x- and y-axes, and call them u_1 and u_2. This then specifies the vector **u** *with respect to* our coordinate system. Suppose there is another scientist (observer S′), stationary relative to S, who chooses his Cartesian axes x' and y' in some different orientation. He also measures the components of the vector **u** with respect to his axes and finds them to be u_1', u_2' and he denotes the vector by **u**′. We want to see how the two sets of components are related to each other. After all, both of them want to describe the same vector, and communicate with each other. Since we are describing the simplest situation, we will treat only the case of observers S and S′ stationary relative to each other.

Since a change of origin does not affect the components of a vector, we concentrate only on the relative orientation of the axes. Suppose the axes x', y' are obtained by a rotation of x, y through an angle θ about the origin, as shown in *Figure* 4. We draw the vector **u** with its tail at the common origin O. The line segment OA is the vector under study. Let AP and AQ be the perpendiculars drawn from the head of the vector on the x-axis and x'-axis, respectively. Then it is clear that

$$OP = u_1\,,\ PA = u_2\,,$$
$$OQ = u_1'\,,\ QA = u_2'. \qquad (11)$$

Now we want to see how these can be related to each other. The angle between

Figure 4. Rotation of a Cartesian coordinate system (x, y) through θ to (x′, y′).



x-axis and x'-axis, as well as between y-axis and y'-axis is θ, and so also is the angle between lines AP and AQ. Let PS be the perpendicular from P on the x'-axis. We can break up OQ into two parts, OS + SQ. Similarly we extend line AQ a little further and draw a perpendicular PR from P to the line AQ, so that AQ = AR – QR. From the simple geometry of right-angled triangles, we see on using (11) that

$$u_1' = u_1 \cos\theta + u_2 \sin\theta,$$

$$u_2' = u_1 \sin\theta + u_2 \cos\theta, \qquad (12)$$

Now we can see, from (12), that

$$u_1'^2 + u_2'^2 = u_1^2 + u_2^2, \qquad (13)$$

which expresses the fact that the magnitude of the vector remains the same for the two observers, though the components of the vector differ for both of them.

Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be two vectors as observed by S. Let the second observer S′ measure their components and denote them by $\mathbf{u}' = (u_1' + u_2')$ and $\mathbf{v}' = (v_1' + v_2')$, respectively. The components v_1', v_2' will be related to v_1, v_2 by relations similar to those in (10), that is

$$v_1' = v_1 \cos\theta + v_2 \sin\theta,$$

$$v_2' = -v_1 \sin\theta + v_2 \cos\theta. \qquad (14)$$

Their conventional scalar product, in two-dimensional Cartesian system, is defined as

$$\mathbf{u}.\mathbf{v} = u_1 v_1 + u_2 v_2, \quad \mathbf{u}'.\mathbf{v}' = u_1' v_1' + u_2' v_2' \qquad (15)$$

Now from (12) and (14), we can see that

$$\mathbf{u}'.\mathbf{v}' = u_1' v_1' + u_2' v_2' = u_1 v_1 + u_2 v_2 \equiv \mathbf{u}.\mathbf{v}, \qquad (16)$$

showing that the scalar product, as defined above, is invariant with respect to the coordinate transformation considered here.

On the other hand, suppose we were to define the scalar product of vectors as

$$\mathbf{u}.\mathbf{v} = au_1v_1 + bu_2v_2, \tag{17}$$

where a and b are constants, and a similar expression for $\mathbf{u}'.\mathbf{v}'$, then using (12), we would get

$$\begin{aligned}\mathbf{u}'.\mathbf{v}' &= au_1'\,v_1' + bu_2'\,v_2' \\ &= \cos^2\theta\,(au_1v_1 + bu_2v_2) + \sin\theta\cos\theta\,(a-b) \\ &\quad (u_1v_2 + u_2v_1) + \sin^2\theta\,(bu_1v_1 + au_2v_2).\end{aligned} \tag{18}$$

It is clear that this is very different from (17), and reduces to it *if and only if* $b = a$. In that case, we may as well choose $a = 1$, so that we have the conventional scalar product as in (15). It is not for nothing that the scalar product of two vectors is defined this way. The above simple transformation shows that only this form remains invariant under coordinate transformations. This is the *only* way to produce a scalar from two given vectors involving both of them linearly.

The vector product of two vectors in the three-dimensional space is conventionally defined as

$$\mathbf{u}\times\mathbf{v} = \mathbf{i}(u_2v_3-u_3v_2) + \mathbf{j}(u_3v_1-u_1v_3) + \mathbf{k}(u_1v_2-u_2v_1), \tag{19}$$

where **i**, **j**, **k** have been defined earlier. Here again, one can see that the transformation of the coordinate system does not change the form of the above expression. This is the *only* way to produce a vector from two given vectors involving both of them linearly. Although we have used the simplest coordinate transformation above, a rotation of the two-dimensional Cartesian coordinate system, this essentially illustrates the concept of invariance.

Physical Laws and Mathematical Equations

The imposition of invariance for scalars and vectors is a very important aspect. Thus if someone were to say that his experiments have shown that the relation between the applied electric field **E** and the current density **j** in a new anisotropic crystal is of

Mathematical equations representing physical laws are postulated on the basis of a combination of experiments and the logic of invariance.

the form $E_1 + 2E_2 - E_3 = \sigma\,(2j_1 - j_2 + j_3)$, where E_i and j_i are the components of **E** and **j** respectively, and σ is a constant, we would immediately reject it as being an unphysical expression or equation. It is possible, as a freak chance, that this relation holds good in one particular coordinate system and for particular values of the applied electric field E_i, but it would not be true in any other coordinate system and for every electric field.

Mathematical equations representing physical laws are postulated on the basis of a combination of experiments and the above logic. Some examples are given here :

(a) ***Work done by a force on an object in displacing it:***

Experiments show that the increase in energy of the object acted upon by a force is proportional to the magnitudes of the applied force **F** and the resulting displacement **s** provided the directions of both **F** and **s** remain unaltered in the experiment. Thus work W is a *linear function* of **F** and **s** and does not involve any higher powers. There is no other parameter. We also note that work is a scalar quantity. The only scalar quantity that can be generated from **F** and **s** and which is linear in both, is **F**.**s**. Thus we get the formula

$$W = k\,\mathbf{F}.\mathrm{s}, \tag{20}$$

where k is a constant. Of course, scientists searched for the logic to match the experience. However, the beauty is that this logic works in a general case.

(b) ***Force on a charge moving in an electric field:***

Experiments show that the force $\mathbf{F}_e$ on an electric charge q in an electric field **E** is proportional to the charge q and also to the applied electric field **E**, and it acts in the direction of **E**. It is independent of any other parameter such as mass or velocity of the charged particle. Force is a vector quantity. The only vector that can be formed with scalar q and vector **E** involving both of them linearly, is $q\mathbf{E}$, and hence

$$\mathbf{F}_e = k\, q\, \mathbf{E}\,, \tag{21}$$

where k is a constant.

(c) ***Force on a charge moving in a magnetic field:***

Experiments show that the force $\mathbf{F}_m$ acting on an electric charge in a magnetic field **B** is proportional to the charge q and to the magnitude of the magnetic field **B**. They further show that the force is proportional to the magnitude v of velocity for a given set of **B**, **v** directions. No other parameter is involved. The only vector that can be constructed from q (a scalar), and **v** and **B** (vectors) involving all three of them linearly is $q\mathbf{v} \times \mathbf{B}$, and hence

$$\mathbf{F}_m = k\, q\, \mathbf{v}\times\mathbf{B}, \tag{22}$$

where k is again a constant. The dependence of $\mathbf{F}_m$ on the sine of the angle between **v** and **B** follows from the logic of invariance, and experiments confirm this. The only choice left to the experiments is to decide between $\mathbf{v}\times\mathbf{B}$ and $\mathbf{B}\times\mathbf{v}$, assuming that the sign convention for cross product of vectors is already decided. It must be understood that the direction of this force is not easy, though not impossible, to determine from experiments. It comes out from the logic of invariance as above, and once we have (22), it can be corroborated by further experiments.

The value of the constant k in each case depends on the choice of units, and is decided by experiments. It may be noted that a combination of (21) and (22) gives the Lorentz force on a charge in combined electromagnetic fields. A generalization of scalars and vectors is the concept of tensors [6]. However, we do not intend to introduce any advanced concepts in this article.

Suggested Reading

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[5] R B McQuistan, *Scalar and Vector Fields*, John Wiley & Sons, New York, 1965.
[6] A W Joshi, *Matrices and Tensors in Physics*, New Age International, New Delhi, 3rd Ed.,1995.