Classroom

In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Plotting Some Interesting Surfaces

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Introduction

One can use software like Mathematica, Matlab or Maple to plot a curve or a surface in $\mathbf{R}^3$ whose equations are known. But, since there is no general method to find equations of a given surface analytically, we found it interesting to consider some well known surfaces to find their parametric equations analytically. Here, in fact, we model those surfaces by exploiting their structural symmetry. We also provide their plots using Mathematica. We find that vector algebra is particularly convenient for finding equations.

Spiral

Let us start with a very simple surface. Let a stick of length 8 units rotate about its mid point on a horizontal plane and go up simultaneously. This stick will trace a path which is certainly a surface. To get its equations in parametric form we take

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$$X = t\cos\theta,$$

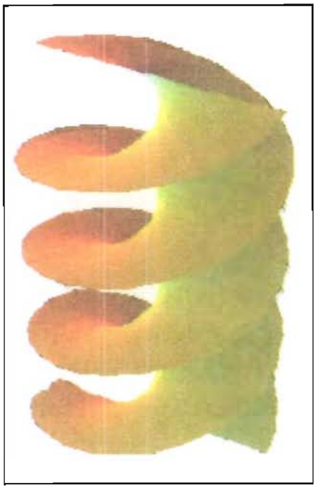


Figure 1. Spiral.

$$Y = t\sin\theta,$$

$$-4 \leq t \leq 4;$$

where, (X, Y) is the parametric form of a circle of radius $|t|$. Since rotation is accompanied by going up, we can take

$$Z = \theta, \text{ where } \theta \in [0, 2n\pi].$$

This gives a parametric equation of the spiral.

A Surface of Revolution

Let us turn to our first general method of plotting surfaces. There is a way to get a surface generated by revolving a plane curve about an axis lying on the plane of the curve. This type of surface is called *a surface of revolution.*

As an example, let us consider the curve

$$\begin{aligned} X &= 0, \quad Y = \sin\theta(1+\cos\theta), \\ Z &= -\cos\theta(1+\cos\theta), \end{aligned}$$

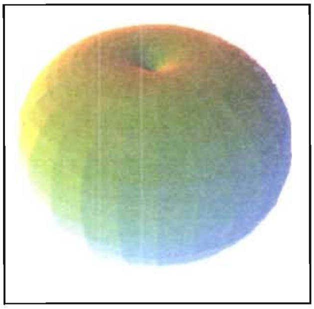
where $\theta \in [0, 2\pi]$.

This curve is called the *cardioid.* Here it is on the Y-Z plane with Z-axis as its axis. Now if we rotate it about Z-axis, we will get a surface. Here, the distance of each point from Z-axis (*i.e.*$\sqrt{X^2+Y^2}$) will be same as $|Y|$ of that curve. So, the equation of the surface is

$$\begin{aligned} \sqrt{X^2+Y^2} &= |\sin\theta(1+\cos\theta)|, Z = -\cos\theta(1+\cos\theta), \\ \text{i.e,} \qquad X &= \sin\varphi \ \sin\theta(1+\cos\theta), \\ Y &= \cos\varphi \ \sin\theta(1+\cos\theta), \\ Z &= -\cos\theta(1+\cos\theta), \end{aligned}$$

where $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$.

Figure 2. Revolution of Cardioid.



Möbius Strip

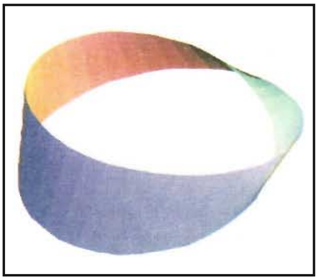


Figure 3. The Möbius Strip.

A Möbius strip is a two dimensional surface obtained by identifying two opposite sides of a rectangle with a flip. We will use vector algebra to find the equation of this surface. Observe that the middle line of a Möbius strip is a circle. The surface can be viewed as a locus of a stick having mid point on that circle and moving along this and also rotating slowly keeping the tangent (at the point of contact) of the circle, as its axis of rotation, in such a way that it can complete only a half rotation while it traverses the whole circle.

Let the center of the circle be vector origin and $\vec{r}(\theta) = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}, \theta \in [0, 2\pi]$ be a point on the circle, where $\hat{i}$ and $\hat{j}$ are unit vectors along x-axis and y-axis respectively.

Let, $\vec{r}(t,\theta)$ be any point on the strip at a distance t from $\vec{r}(\theta)$. So $(\vec{r}(t,\theta) - \vec{r}(\theta))$ is rotating about $\vec{r}(\theta)$ on the plane containing $\vec{r}(\theta)$ and $\hat{k}$, to make a half circle of radius t as θ runs from 0 to 2π.

Therefore,

$$\vec{r}(t,\theta) - \vec{r}(\theta) = t\Big[\cos\frac{\theta}{2}\ \frac{\vec{r}(\theta)}{|\vec{r}(\theta)|} + \sin\frac{\theta}{2}\hat{k}\Big].$$

This implies that

$$\vec{r}(t,\theta) = (1 + t\cos\frac{\theta}{2})\vec{r}(\theta) + t\sin\frac{\theta}{2}\hat{k},$$

where $t \in [-0.3, 0.3]$.

This gives the following equation of Möbius strip

$$X(= \text{coefficientof } \hat{\text{i}}) = \cos\theta(1 + t\cos\frac{\theta}{2}),$$

$$Y(= \text{coefficientof } \hat{\text{j}}) = \sin\theta(1 + t\cos\frac{\theta}{2}),$$

$$Z(= \text{coefficientof } \hat{k}) = t\sin\frac{\theta}{2},$$

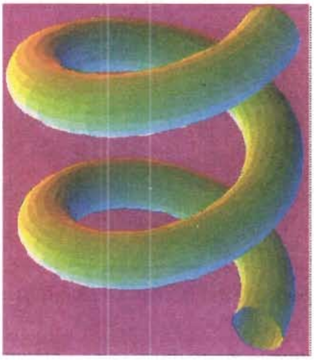


Figure 4. The Pipe.

where $-0.3 \leq t \leq 0.3$ and $0 \leq \theta \leq 2\pi$.

Coiled Cylinder

We can exploit the same idea to find out the parametric representation of another interesting but common surface. We consider a cylinder coiled as a telephone wire as our next surface. Here we shall take

$$\vec{r}(\theta) = a\cos\theta\ \hat{i} + a\sin\theta\ \hat{j} + c\theta\ \hat{k},$$

where a and c are positive real numbers and $\theta \in [0, 2n\pi]$. In fact, it is a point on a spring. Now, as before, we need to understand the equation of $\vec{r}(\varphi, \theta)$. After a little observation one should arrive at the following,

$$\vec{r}(\varphi,\theta) - \vec{r}(\theta) = b\Big[\sin\varphi\ \frac{\vec{r}(\theta) - c\theta\hat{k}}{|\vec{r}(\theta) - c\theta\ \hat{k}|} + \cos\varphi\hat{k}\ \Big].$$

where $b > 0$ and $\varphi \in [-\pi, \pi]$; that is,

$$\vec{r}(\varphi,\theta) = \cos\theta(a + b\sin\varphi)\ \hat{i} + \sin\theta(a + b\sin\varphi)\ \hat{j} +$$

$$(c\theta + b\cos\varphi)\ \hat{k}.$$

Therefore the equation is

$$\begin{aligned} X &= \cos\theta(a + b\sin\varphi), \\ Y &= \sin\theta(a + b\sin\varphi) \quad \text{and} \\ Z &= (c\theta + b\cos\varphi), \quad \text{where } \theta \in [0, 2n\pi],\ \varphi \in [-\pi, \pi]. \end{aligned}$$

Some More Interesting Surfaces

Till now, we have found equations of surfaces which come due to some mathematical considerations. Now we are going to find the equation of some interesting surfaces which occur in our daily life. For that, besides the mathematical methods we need some intuition also.

Pitcher

We want to make a surface like a pitcher. To obtain it as a surface of revolution, the equation of the curve we choose is

$$y^2 - x^3 + 4x = 5.$$

Then the equation of the pitcher comes out to be,

$$\begin{aligned} X &= \left(\sqrt{t^3 - 4t + 5} - 0.3\right)\sin\theta, \\ Y &= \left(\sqrt{t^3 - 4t + 5} - 0.3\right)\cos\theta, \\ Z &= 1.3t, \end{aligned}$$

where $\theta \in [0, 2\pi]$ and t $\in [-2.45, 1.8]$.

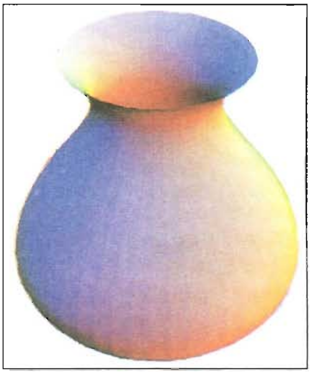


Figure 5.The Pitcher.

One may ask "where are so many numerical constants coming from?" Simply speaking, these come after some trial and error to make the pitcher good looking and complete.

Surface of Lamp

Let $\Gamma(x)$ be the extended gamma function. Then the function $\frac{1}{\Gamma(x)}$ has a very interesting graph when x is negative. We use this curve to find an equation of the surface of a lamp as a surface of revolution. The set of equations we found is

$$\begin{aligned} X &= \left(2 + \frac{1}{\Gamma(t)}\right)\cos\theta, \\ Y &= \left(2 + \frac{1}{\Gamma(t)}\right)\sin\theta \quad \text{and} \\ Z &= t, \end{aligned}$$

where $t \in [-4, 8]$, $\theta \in [0, 2\pi]$.

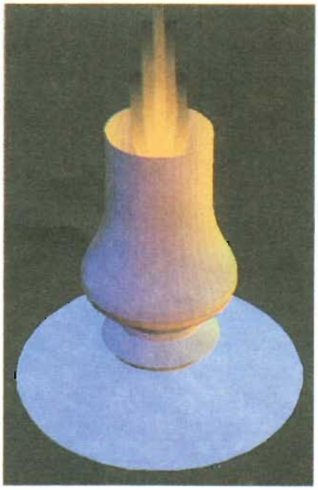


Figure 6. The Lamp.

Surface of Heart

Now, we consider the most interesting example - the surface of the heart. Here we shall try to understand how a simple parametric equation can represent the surface of heart. To do that, observe the following:

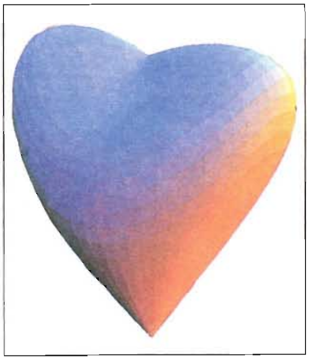


Figure 7. The Heart.

1. The vertical line joining the bottom and the saddle point at the top can be considered as the Z-axis and there is some sort of symmetry about this Z-axis.

2. Any vertical plane through the Z-axis will intersect the surface in a closed curve. And those closed curves are symmetric about the Z-axis. So, let us divide such a closed curve into two arcs such that they look like mirror images of each other about the Z-axis.

3. Observe that the surface may be viewed as a trajectory of those curves; that means if we can find the equation of each arc, then we can find the equation of the surface also. Let us call the family of the arcs as $\{\Gamma_\theta\}$ such that, as θ runs from 0 to 4π, the plane of arcs Γ_θ will rotate from angle 0 to 2π uniformly. So, two opposite arcs will be Γ_θ and $\Gamma_{\theta+2\pi}$. Let the parametric equation of Γ_θ be

$$X = X(\theta, t),\ Y = Y(\theta, t),\ Z = Z(\theta, t);$$

$$\theta \in [0, 4\pi],\ t \in [0, 1].$$

4. To find the parametric equation of Γ_θ for some θ in $[0, 4\pi]$, we need to observe how exactly the curves Γ_θ evolve as θ changes. Consider the left most arc Γ_0. We note that each and every point of Γ_θ is shifted down continuously as θ goes from 0 to π (meaning rotation by $\pi/2$) to get different Γ_θ's. After that all the points shift up continuously till we get $\Gamma_{2\pi}$ which is the right most arc (just opposite to Γ_0). This going down and up is repeated as θ changes from 2π to 4π, and finally we get $\Gamma_{4\pi} (= \Gamma_0)$.

5. Let all the arcs Γ_θ start from the bottom $(-2, 0, 0)$ and end at the top $(0, 0, 0)$. But, observe that, in between, all are blowing away from Z-axis. So,

$$\begin{aligned} X(\theta, 0) &= Y(\theta, 0) = 0 \\ X(\theta, 1) &= Y(\theta, 1) = 0 \text{ for all } \theta. \end{aligned}$$

$$\text{and for } t \in (0,1)\ X(\theta,t) > 0$$
$$Y(\theta,t) > 0 \text{ for all } \theta.$$

So we can take

$$\begin{aligned} X(\theta,t) &= \sin\pi t\, g_1(\theta), \\ Y(\theta,t) &= \sin\pi t\, g_2(\theta), \end{aligned}$$

where the functions g_1, g_2 are positive.

Again, assume that as θ goes from 0 to 4π for fixed t, the point $(X(\theta,t), Y(\theta,t))$ rotates through an angle of 2π along some elliptic path. We choose the ellipse to be

$$X^2 + \frac{Y^2}{(0.7)^2} = 1.$$

We could have chosen a circular path $X^2 + Y^2 = 1$; this would result in a similar type of surface but not exactly the shape that we want. This 0.7 was found after several trials.

$$\text{Then,}\quad \begin{aligned} X(\theta,t) &= \sin(\pi t)\cos\frac{\theta}{2} \text{ and} \\ Y(\theta,t) &= 0.7\sin(\pi t)\sin\frac{\theta}{2}. \end{aligned}$$

6. As we have mentioned in 5, we shall take

$$Z(\theta,t) = 2\Big[P_t\Big((0,-1),(\tau(\theta),f(\theta)),(1,0)\Big)\Big],$$

where $P_t\Big((x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\Big)$ is a polynomial in t of degree at most $n-1$ which takes the value y_i at $t = x_i; i = 1,2,\ldots,n$. Then by using Newton's divided difference formula, we get

$$Z(\theta,t) = 2\Big[-1 + t\,\frac{1+f(\theta)}{\tau(\theta)}$$
$$+\, t(t-\tau(\theta))\frac{\tau(\theta)-(1+f(\theta))}{\tau(\theta)(1-\tau(\theta))}\Big].$$

In fact, we shall take the functions $\tau(\theta)$ and $f(\theta)$ in such a way that the point $(\tau(\theta), f(\theta))$ goes down for $0 < \theta < \pi$ and goes up for $\pi < \theta < 2\pi$, again goes down for $2\pi < \theta < 3\pi$ and goes up for $3\pi < \theta < 4\pi$. For that we choose $f(\theta) = \frac{\cos\theta - 1}{4}$, whose range is $[-1/2, 0]$. We may take $\tau(\theta)$ as 0.5 (constant). But for the sake of a good looking surface we take $\tau(\theta)$ as a function of θ and taking values from 0.4 to 0.6; that is

$$\tau(\theta) = 0.6 + 0.4 f(\theta).$$

After substituting the expressions of $f(\theta)$ and $\tau(\theta)$ in the expression for $Z(\theta, t)$ and simplifying, we get,

$$Z(\theta, t) = \frac{(t-1)(-49 + 50t + 30t\cos\theta + \cos 2\theta)}{(-25 + \cos^2\theta)}.$$

So, we get explicitly functions $X(\theta, t), Y(\theta, t)$ and $Z(\theta, t)$ and the equation of arcs and hence the equation of surface also. That is,

$$\begin{aligned} X &= \sin(\pi t)\cos\frac{\theta}{2}, \\ Y &= 0.7\sin(\pi t)\sin\frac{\theta}{2} \quad \text{and} \\ Z &= \frac{(t-1)(-49 + 50t + 30t\cos\theta + \cos 2\theta)}{-25 + \cos^2\theta}, \end{aligned}$$

where $\theta \in [0, 4\pi],\ t \in [0, 1]$.

To conclude, we hope these plots will get the reader interested in modelling surfaces in an analytic way. We have tried to keep the ideas straightforward so that they are easily understandable.