Motion of Charged Particles in Electromagnetic Fields and Special Theory of Relativity

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The electromagnetic interaction governs many aspects of our daily lives. Although the unity of the electric and the magnetic phenomena is established through Maxwell's theory, it is not adequately emphasized in most of the courses on electrodynamics that the electric interaction together with the special theory of relativity provides a firm basis for magnetism, which follows as a natural consequence. In this article is reported a computer simulation of trajectories of charged particles in electromagnetic fields as observed from different inertial frames of references. An examination of these trajectories offers a vivid illustration of charged particle dynamics in electromagnetic fields and reveals the relationship between the electromagnetic interaction and the special theory of relativity.

1. Introduction

The easy availability of desktop computers in academic institutions is bringing about a rapid revolution in physics teaching curricula all over the world. This revolution is affecting not only 'how' physics is taught, but also 'what' physics is taught [1]. Some physics teaching-aid software development programs have already been undertaken in several institutions and are implemented with great success. Excellent books on computational physics which address this task have also been published in recent years[1-5]. Here, at the IIT-Madras, we have recently begun a program involving some undergraduate and postgraduate students in an effort to develop physics educational software. The program is called CAPE-IITM: Computer Aided Physics Education at the Indian Institute of Technology, Madras. In

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this article, we describe the simulation of trajectories of charged particles as seen from different inertial frames of reference. More importantly, the solutions are then used to reveal an important aspect of the relationship between the electromagnetic interaction and the special theory of relativity.

2. Charged Particle Dynamics in Electromagnetic Fields

We had the following objectives in mind in undertaking this problem:

1. It is useful for students to see how a point charge moves [6-11] under the influence of an electromagnetic field($\vec{E}$, $\vec{B}$).

2. In relativistic mechanics, the directions of force and acceleration are not necessarily parallel, and the usual interpretation of mass (inertia) as the proportionality between force and acceleration breaks down[8,10].

A clear understanding of these subtleties is important.

The Lorentz force on a charge q, moving at a velocity $\vec{v}$ in an inertial frame of reference in which the electromagnetic field is ($\vec{E}$,$\vec{B}$), is given by q ($\vec{E}$ + $\vec{v}$ x $\vec{B}$). This is approximated to mass times acceleration only at low velocities. The correct law, valid at any velocity must identify the force $\vec{F}$ with $\frac{d\vec{p}}{dt}$, where $\vec{p}$ is given by $\vec{p} = \gamma\, m_o \vec{v}, \gamma$ being $1/\sqrt{(1 - v^2/c^2)}$, c being the speed of light, and m_o the rest mass of the particle. We note that the expression q ($\vec{E}$ + $\vec{v}$ x $\vec{B}$) represents a certain consideration of electric and magnetic forces on a charge q in a given *inertial* frame of reference S. In a different *inertial* frame of reference S' moving with respect to the frame S, the expression for this electromagnetic force has *different* electric and magnetic components($\vec{E'}$ $\vec{B'}$). The Lorentz force must now be constructed in terms of the *primed* quantities.

It is important to recognize that the electric and magnetic forces are part of a single physical phenomenon: electromagnetic interaction. The detailed separation of this interaction into electric and magnetic parts depends on the *specific* frame of reference chosen to describe the interaction, but the overall electromagnetic description remains invarinat according to Einstein's special theory of relativity[8,10].

To demonstrate this feature, we have developed a software in which the equation of motion

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} \qquad \text{where} \qquad \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}), \tag{1}$$

is solved for various initial conditions of the values of q, m_o, $\vec{E}, \vec{v}$ and $\vec{B}$ to be chosen interactively by the user in the menu driven program.

In relativistic mechanics, one must employ [9,10]

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} + m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}, \tag{2}$$

where m $= \gamma m_o$. Accordingly, equation (1) takes the following form

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\vec{a} = \vec{F} - \frac{\vec{v}}{c}[\vec{F}.\frac{\vec{v}}{c}] \tag{3}$$

Equation (3) shows that the acceleration coincides with the direction of the force only when the net force $\vec{F}$ is perpendicular to the velocity $\vec{v}$, unless one makes the non-relativistic approximation that the speed of light is infinite. In all other cases, there is an additional term involving an acceleration in the direction of velocity, albeit smaller by a factor of $\frac{v^2}{c^2}$, relative to that in the direction of the force. The program we have developed numerically solves (3) using (1) by slicing the time of investigation into very small intervals δt. The position $\vec{r}_{i+1}$ and the velocity $\vec{v}_{i+1}$ at the beginning of the$(i+1)$th interval are obtained from $\vec{r}_i$ and $\vec{v}_i$ using the following relations:

$$\vec{r}_{i+1} = \vec{r}_i + \delta t \vec{v}_i + \frac{1}{2}\delta t^2 \vec{a}_i, \tag{4}$$

and

$$\vec{v}_{i+1} = \vec{v}_i + \delta t \vec{a}_i, \tag{5}$$

$\vec{a}_i$ being obtained from equation(3). For example at t=0 which is to be understood as the beginning of the zeroeth interval, the values of q, $\vec{E}$ $\vec{B}$ and the initial value of $\vec{v}$ (say $\vec{v}_0$), are used in equation(1) to get the value of $\vec{F}$ at t=0 (say $\vec{F_0}$). This value $\vec{F_0}$ is then used along with $\vec{v_0}$ in equation(3) to obtain the value of the acceleration $\vec{a_0}$ at t = 0. Thus knowing $\vec{r_0}$ the initial position, $\vec{v_0}$ and $\vec{a_0}$, the position $\vec{r_1}$ and velocity $\vec{v_1}$ at the beginning of the first interval, which is at t = δt, are obtained from (4) and (5) respectively. This procedure is continued iteratively to obtain the position and velocitiy of the particle at various subsequent times.

Results for the trajectory of the charged particle q are discussed in the next section for different values of ($\vec{E}$,$\vec{B}$) This part of the software is by itself rather instructive to see on the computer monitor, since a laboratory experiment to see the response of a charged particle to an electromagnetic field is a formidable task.

The next task our software addresses is the examination of these trajectories from *another* inertial frame of reference S', which moves with respect to S at a constant velocity $\vec{v_f}$ along the X-direction. To get the trajectory of the charged particle in the frame S' one can employ two *independent* and *alternate* procedures:

(i) Point Transformation Technique: One can use the Lorentz transformations from ($\vec{r}$,t) to ($\vec{r}$',t') to get the trajectory $\vec{r}$' = $\vec{r}$'(t') of the particle in the frame S', where

$$x' = \frac{x - v_f t}{\sqrt{(1 - \frac{{v_f}^2}{c^2})}}, \qquad y' = y, \qquad z' = z \tag{6}$$

and

$$t' = \frac{t - \frac{v_f}{c^2}x}{\sqrt{1 - \frac{v_f^2}{c^2}}}. \qquad (7)$$

(ii) Field Transformation Technique: Alternatively, one can use the transformation laws for the electric and magnetic fields ($\vec{E}$ $\vec{B}$) as one goes from the frame S to S' and *then* solve the equation of motion all over again in the primed frame S' using the numerical algorithm already described above. The transformation equations for the electric and magnetic fields are given by

$$E_x' = E_x \qquad B_x' = B_x \qquad (8)$$

$$E_y' = \gamma_f[E_y - v_f B_z] \qquad B_y' = \gamma_f[B_y + (\frac{v_f}{c^2}E_z)] \quad (9)$$

$$E_z' = \gamma_f[E_z - v_f B_y] \qquad B_z' = \gamma_f[B_z + (\frac{v_f}{c^2}E_y)] \quad (10)$$

where γ_f is $\frac{1}{\sqrt{(1-\frac{v_f^2}{c^2})}}$. In the next section, we present some results of the above mentioned two alternative procedures as implemented in our software.

3. Charged Particle Trajectories

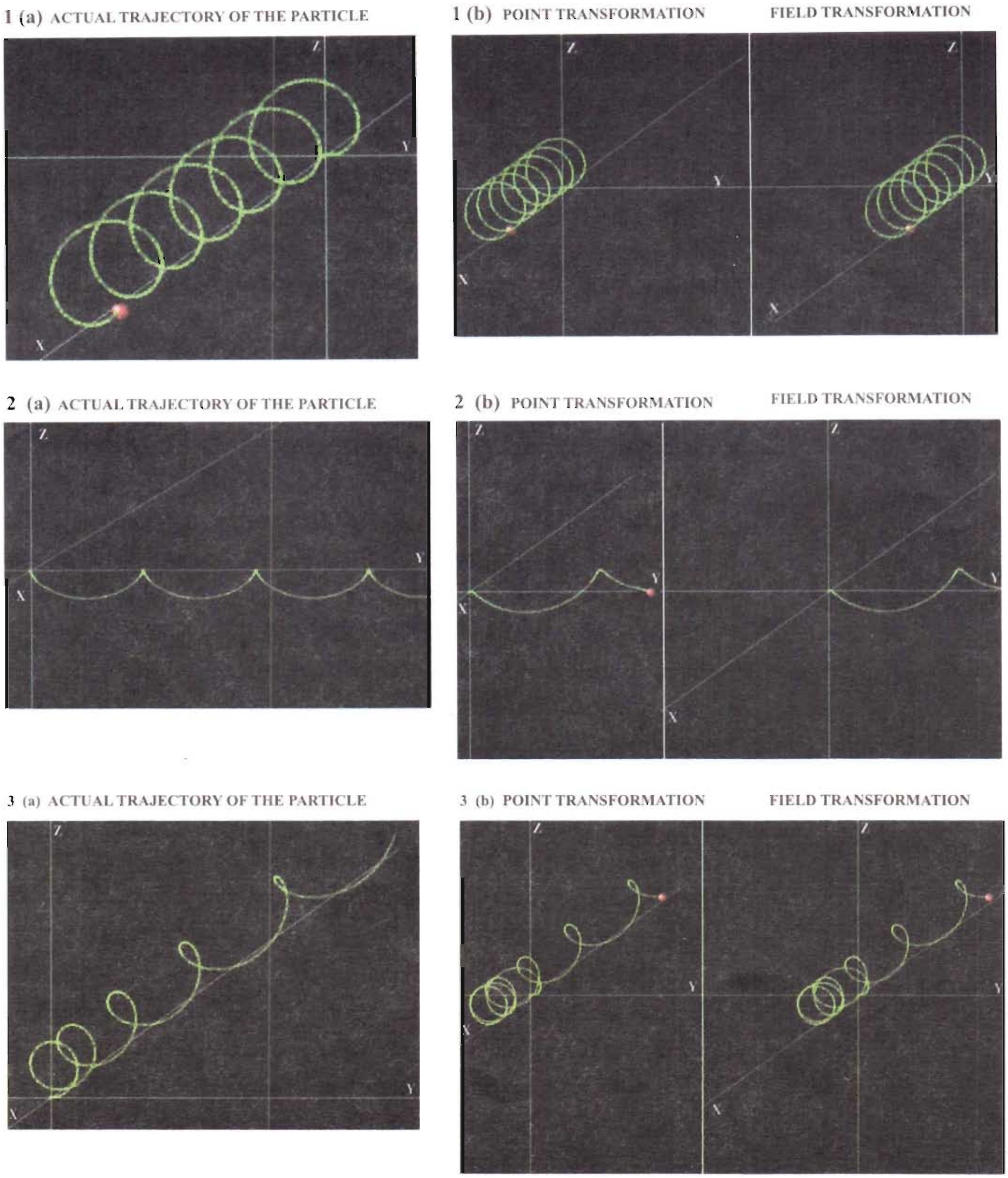
In the simulation program we have developed, the user can select the values of the fields $\vec{E}$ and $\vec{B}$ the charge q and the rest mass m_o of the particle and also its initial conditions (i.e., its initial position and velocity). We present here a few illustrations:

Figure 1(a) shows the path of an electron in a magnetic field of $0.1\hat{e}_x$ Wb/m^2 when projected from the origin of a Cartesian coordinate system with a velocity of (4.6 x $10^7\hat{e}_x$+ 2.65 x $10^8\hat{e}_y$) m/s. A time step δt of 10^{-14} sec was used in (4) to get the result shown here. As expected, the trajectory is a helix. In the software that we have developed, the trajectory of the charged particle as seen by an observer in the frame S' is calculated *independently* using the two procedures mentioned in

Section 2. The results are then depicted side by side in adjacent windows on the monitor(*Figures* 1(b)(L) and 1(b)(R)). For an observer in S' moving with a velocity 2.25 x $10^7\hat{e}_x$ m/s, the trajectory will appear to be a helix with a pitch that is seen to be distinctly less compared to that of the helix seen by the observer in frame S. The two figures 1(b)(L) and 1(b)(R) are completely congruent, as expected. An examination of the actual numerical values shows that the difference in the position of the particle obtained using the two procedures, in the present case, is less than 2 percent.

Figure 2(a) shows the trajectory of an electron initially at rest at the origin of the frame S in transverse electric and magnetic fields ($\vec{E} = 10\hat{e}_z kV/m$ and $\vec{B} = 0.05\hat{e}_x Wb/m^2$) as seen by an observer in S. To an observer in a frame S' moving with a constant velocity 7.5 x $10^4\hat{e}_x$ m/s with respect to S, the trajectory will appear to be as shown in figures 2(a)(L) and 2(a)(R), both of which are once again congruent with each other.

Figure 3(a) shows the trajectory of an electron initially at the origin of S with the electric field $35\hat{e}_x$ kV/m and a magnetic field of 0.05 $\hat{e}_x$ Wb/m^2. The electron is projected at an initial velocity of 2.65 x $10^7\hat{e}_y$m/s. The electric field accelerates the electron in the x-direction and the magnetic field causes the particle to rotate (as in case1). The resulting trajectory is therefore a helix with increasing pitch. An observer fixed to a frame S', which is moving with a velocity of (-1.0 x $10^7\hat{e}_x$) m/s with respect to the frame S, will see the particle initially move backwards in the S' frame, but later on reverse its direction and then catch up with him(her) and subsequently overtake him(her) as shown in figures 3(b)(L) and 3(b)(R). The trajectories obtained by both (i) the point transformation technique and (ii) the field transformation technique, are once again congruent, as can be seen from *Figures* 3(b)(L) and 3(b)(R).



Figures 1-3. Illustrations of trajectories of an electron in different frames of reference. See text for details.

One person's electric field is another's magnetic field.

4. System Requirements

Since the software developed was coded in Java, the programs are expected to be platform independent. The users need Java2 or higher to be installed on their machines. Java2 is available for free from Sun Microsystems. Since the graphics can be demanding on the system memory, we recommend that the programs be run on a system that has atleast 128MB of RAM and a standard desktop configuration for everything else.

5. Concluding Remarks

The software we have developed is 'user friendly', and besides depicting the experimentally formidable task of showing particle trajectories in electromagnetic fields, it illustrates the intimate dependence of the electric and magnetic components of the electromagnetic interaction on the choice of the particular frame of reference that is employed to describe the interaction. The software further brings out the interconnections between the electrodynamic interaction and the special theory of relativity. The transformation relations (equations (7a), (7b) and (7c)) from $(\vec{E}, \vec{B})$ to $(\vec{E}', \vec{B}')$ employ linear superpositions of the components of the electric and magnetic fields and thus bring out the unity of electricity and magnetism, perhaps even more forcefully than Maxwell's equations which relate vector functions of $\vec{E}$ with those of $\vec{B}$, and vice-versa. Surely, a linear superposition manifestly expresses the fact that the electric and magnetic fields are two expressions of essentially the same physical entity. This fact is most elegantly stated by this quotation from D J Griffiths' *Introduction to Electrodynamics* which goes as "...one person's electric field is another's magnetic field". We trust that this user-friendly software helps elucidate crucial fundamental concepts in electrodynamics and special theory of relativity.

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Suggested Reading

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