In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

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Power Maps and Commutativity of Groups

Some of the interesting problems in an undergraduate course on group theory involve the power maps $g \mapsto g^n$ for some n. Two standard results found, for instance, in Herstein [1] are:

Problem 3, page 35. *If G is a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is Abelian.*

Problem 24, page 48. *If G is a finite group such that 3 does not divide $|G|$ and $(ab)^3 = a^3b^3$, then G is Abelian.*

In view of the above two results, it is natural to ask:

Let $p > 3$ be a prime and G be a finite group such that p does not divide $|G|$. If the p-th power map is a homomorphism on G, (that is, if $(ab)^p = a^pb^p$, for all $a, b \in G$), then, is G necessarily Abelian ?

The following example shows that the answer to this question is in the negative.

1. Example

Let $p > 3$ be a prime. Let $n = p - 1$. Let D_n be the dihedral group of order $2n = 2p - 2$. Recall that it is generated by a, b satisfying $a^2 = 1, b^n = 1, aba^{-1} = b^{-1}$.

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It can also be realised as the group of symmetries of a regular n-gon. Since $n > 2, D_n$ is not Abelian. Since half of the elements of this group are of order 2 and the other half form a cyclic group of order n and n is even, we get $x^n = 1$ for all $x \in G$. Thus $x^p = x$ for all $x \in G$. So, trivially $(xy)^p = x^p y^p$ for all $x, y \in D_n$.

Thus, the answer to our question is in the negative for $p > 3$. For $p > 5$, we can find an even smaller sized counter example. Suppose $p - 1 = 2m, m \geq 3$. Let $G = D_m$ be the dihedral group of order $2m$. G is not Abelian. Clearly $x^{p-1} = 1$ for all $x \in G$. So $(xy)^p = x^p y^p$ for all $x, y \in G$.

Contrast this with another result (Problem 15, page 103, in [1]) which asserts:

If p is a prime dividing $O(G)$ and the p-th power map is a homomorphism on G, then G has nontrivial center and a unique p-Sylow subgroup P and a normal subgroup N such that $G = PN$ and $P \cap N = 1$.

We prove the following result on the positive side.

2. Theorem

Let G be a finite group and for each natural number n, let θ_n denote the map $g \mapsto g^n$ on G. Then, we have:

(i) *If $(n - 1, |G|) = 1$, then θ_n has no fixed points on G other than the identity.*

(ii) *If θ_n is a homomorphism, then $x^n y^{n-1} = y^{n-1} x^n \quad \forall x$, $y \in G$. Further if also $(n - 1, |G|) = 1$, then θ_{n-1} is a bijection on G,*

(iii) *θ_n is an automorphism if, and only if, it is a homomorphism and $(n, |G|) = 1$, and*

(iv) *If θ_n is an automorphism, then $x^{n-1} \in Z(G)$, the centre of $G \quad \forall x \in G$. If also $(n - 1, |G|) = 1$, then G is Abelian.*

PROOF. (i) Evidently, $\theta_n(x) = x$ implies that $x^{n-1} = 1$ and hence $x = 1$.

(ii) Now $(ab)^n = a^n b^n$ implies

$$(ba)^{n-1} = a^{n-1} b^{n-1}. \qquad (1)$$

Premultiplying by ba we get $(ba)^n = ba^n a^{n-1}$. So $b^n a^n = ba^n b^{n-1}$, which is $b^{n-1} a^n = a^n b^{n-1}$. Now suppose that in addition to θ_n being a homomorphism, we are also given that $(n-1, |G|) = 1$. If $x^{n-1} = y^{n-1}$, we have, by (1), $(y^{-1}x)^{n-1} = x^{n-1}(y^{-1})^{n-1} = x^{n-1}(y^{n-1})^{-1} = 1$. As $(n-1, |G|) = 1$, we get $y^{-1}x = 1$. Thus, θ_{n-1} is 1–1 and hence onto.

(iii) Clearly, if $(n, |G|) = 1$, then θ_n is 1–1 and hence, onto as well. Conversely, if θ_n is an automorphism and p a prime dividing $(n, |G|)$, then there is an element of order p, by Cauchy's Theorem (see Theorem 2.11.3, page 87, in [1]) and this element is in Ker θ_n.

(iv) Suppose θ_n is an automorphism. We know from (ii) that $x^n y^{n-1} = y^{n-1} x^n \;\; \forall x, y \in G$. Since θ_n is a bijection, so every element of G is an nth power. So $y^{n-1} \in Z(G) \;\; \forall y \in G$. Now supose also that $(n-1, |G|) = 1$. Then, by (ii), θ_n is a bijection. So every element of G is an $(n-1)$th power. So $G \subseteq Z(G)$. Hence G is Abelian.

3. Remarks

(a) The condition $(n-1, |G|) = 1$ in (iv) is necessary. For example, if $n = |G| + 1$, then θ_n is the identity map but there are certainly nonabelian groups !

(b) If n has prime order in the multiplicative group of integers mod $|G|$, and $(n-1, |G|) = 1$ and θ_n is a homomorphism, then, by Theorem 2(ii), it is a fixed-point-free automorphism of prime order and a powerful theorem of J G Thompson [2] asserts that G must be nilpotent. However, by Theorem 2(iv), G is in fact Abelian. A very interesting related result is the following (Problem 11, page 70 in [1]):

If an automorphism θ of a finite group G has no fixed points other than 1, and has order 2, then G is Abelian.

We prove one final commutativity result using power maps.

4. Theorem

Let G be any group in which l-th powers of elements commute among themselves and m-th powers of elements commute among themselves. If $(l, m) = 1$, then G must be Abelian.

Proof. Write $al + bm = 1$ for some integers a, b. As any $g \in G$ is expressible as $g = (g^a)^l (g^b)^m$, it suffices to prove that $x^l y^m = y^m x^l$ for all $x, y \in G$. Now,

$$(x^l y^m)^{al} = x^l (y^m x^l)^{al-1} y^m = x^l (y^m x^l)^{al} (y^m x^l)^{-1} y^m =$$

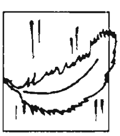
$$(y^m x^l)^{al} x^l (y^m x^l)^{-1} y^m = (y^m x^l)^{al}.$$

Similarly we can show that $(x^l y^m)^{bm} = (y^m x^l)^{bm}$. Multiplying the two equalities, we get the asserted equality.

Suggested Reading

[1] I N Herstein, *Topics in Algebra*, Second Edition, Wiley Eastern Limited & New Age International Limited, New Delhi, 1975.

[2] J G Thompson, Finite groups with fixed-point-free automorphism of prime order, *Proc. Nat. Acad. Sci.*, Vol. 45, pp.578-581, 1959.



Twinkle, twinkle, quasi-star,
Biggest puzzle from afar.
How unlike the other ones,
Brighter than a trillion Suns.
Twinkle, twinkle, quasi-star,
How I wonder what you are!

– *George Gamow*
'Quasar' (1964)

(In commemoration of the discovery of quasars)