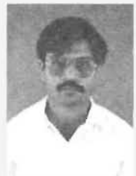
Waring's Problem and the Circle Method

C S Yogananda

In 1770, in his book *Meditationes Algebraicae*, Edward Waring made the statement that *every positive integer is a sum of nine cubes, is also a sum of not more than 19 fourth powers, and so on.* The *so on* was taken to mean that *given a positive integer k there is a number depending only on k, say s, such that every positive integer can be expressed as a sum of at most s number of k-th powers.* There is no obvious heuristic reason to believe the truth or falsity of the statement. There are examples either way. Lagrange had proved, coincidentally in 1770, that every positive integer can be written as a sum of not more than four squares. On the other hand, if one wants to write any positive integer as a sum of powers of 2 then it is not too difficult to see that there is no finite number, say m, such that every positive integer can be written as a sum of m or fewer powers of 2. (Proof: Suppose on the contrary that there is such a number m. But then $2^{m+1} - 1$ can not be written as a sum of m or fewer powers of 2.) But the gut feeling among the mathematicians was that a finite s existed for each k and the smallest such s for a given k came to be denoted by $g(k)$.



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In 1909 Hilbert was able to show that $g(k)$ is finite for every k, i.e., $g(k) < \infty$. But the bound which was deduced from his result was quite high.

Among the formulas which Srinivasa Ramanujan had written in his first letter to G H Hardy was a formula for $p(n)$, the number of partitions of a natural number n. Hardy found it a very striking formula and when Ramanujan joined him in Cambridge this was one of the problems they worked on. Their proof of a slightly weaker form of Ramanujan's original formula was published in 1918. (But as it turned out Ramanujan's was

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"rediscovered" by H Rademacher in 1937. See *Resonance*, Vol.1, No.12, Dec (1996), pp.86-87.) Hardy and Littlewood realised that the method which was used in the proof could be profitably used in the solution of the Waring's problem. This method has come to be known as the *Hardy–Littlewood–Ramanujan circle method.*

Hardy–Littlewood–Ramanujan Circle Method

The starting point is the following integral formula: when n is an integer

$$\int_0^1 e^{2\pi i n x}\, dx = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

This is known as the *counting function* since it can be used to count the number of solutions of equations in integers as, for example, in the present situation. For a given positive integer k, denote the number of representations of a positive integer n as a sum of s number of k-th powers by $r_{k,s}(n)$. Then we have that:

$$r_{k,s}(n) = \sum_{x_1, x_2, \dots, x_s} \int_0^1 e^{2\pi i (x_1 + x_2 + \cdots + x_s - n)}\, dx.$$

Thus we need to determine the existence of an s for which $r_{k,s}(n) > 0$ for all positive integers n. The circle method gives a way of estimating this integral as a function of k and s.

The starting point of the estimation of the above integral formula for $r_{k,s}(n)$ is the division of the interval of integration $[0, 1]$, into *major arc* and *minor arc.* (Even though we are considering the interval and its divisions, since the integration involves points on the unit circle these divisions are referred to as 'arcs'.) Major arc is expected to give the dominant term and the minor arc, the error term. But as it turns out, handling the major arc is fairly simple while it is the minor arc estimation that accounts for the 'major' amount of work involved!

Guided by the theory of modular functions Hardy and Ramanujan suggested that the contribution to the dominant term may come from 'small' intervals around rational numbers with 'small' denominators. The precise values of 'small' in both instances will depend on k and ingenuity in choosing these values correctly is what gives better estimates for s. The union of these 'small' intervals is the major arc and its complement in $[0, 1]$ is the minor arc.

What could be the order of the dominant term? We want to write n as a sum of s numer of k-th powers. Choose $(s-1)$ number of k-th powers less than n which can be done in $n^{(s-1)/k}$ ways and the probability that the remaining number, $n-(x_1+x_2+ \quad +x_{s-1})$, being a k-th power is $n^{(1-k)/k}$. Thus the dominant term is of the order of $n^{(s-k)/k}$. As commented earlier, this is not too difficult to prove.

The task is now to choose s so that the error term is of a lower order. Essentially the only tool that is available to estimate the minor arcs is Weyl's lemma on *uniform distribution of sequences mod* 1:

Lemma (Weyl): *The necessary and sufficient condition for the uniform distribution of fractional parts of the numbers of a sequence of irrational numbers* $f(1)$, $f(2)$, $f(3)$, *in* $(0, 1)$ *is that for every integer* h *we have*

$$\sum_{j=1}^{n} e^{2\pi i h f(j)} = o(n).$$

In their first papers on the subject Hardy and Littlewood were able to get reasonable bounds for $g(k)$.

How does one guess a value for $g(k)$? The idea is to look for numbers which require the most k-th powers among the first few numbers. For $k=2$ it is easily seen that any number of the form $8m+7$ will require 4 squares and so $g(2) \geq 4$, and Lagrange proved in 1770 that $g(2)=4$. After some search we see that 23 will require 9 cubes and 79 will require nineteen 4th powers. Thus one sees that the number $2^k[(3/2)^k] - 1$ will require $[(3/2)^k] - 1$ number of 2^k and $2^k - 1$ number of 1's. Therefore,

$$g(k) \geq 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$$

and it was conjectured that the equality would hold always.

Building on the work of Hardy and Littlewood, S S Pillai and Dickson, independently, were able to show that this is almost the case for $k \geq 6$. The final formula is as follows:

$$g(k) = 2^k + \left[\left(\frac{3}{2}\right)^k\right] - 2$$

provided that

$$2^k \left\{\left(\frac{3}{2}\right)^k\right\} + \left[\left(\frac{3}{2}\right)^k\right] \leq 2^k. \qquad (*)$$

In the case of (*) not holding true also they have given the formula – $g(k)$ will be one less than the term on the RHS of the above formula. But it is conjectured that (*) will hold true for all k. With the determination of $g(k)$ for $k = 5$ by Chen (1964), $g(5) = 37$, and for $k = 4$ by Balasubramanian, Deshouillers and Dress (1986), $g(4) = 19$, the determination of $g(k)$ is complete for all k.

Hardy and Littlewood realised early that more interesting quantity to determine than $g(k)$ would be $G(k)$ which is defined as the smallest number s such that every *sufficiently large* positive integer can be written as a sum of s number of kth powers. Since numbers of the form $8m + 7$ all require 4 sqaures we have $g(2) = G(2) = 4$. Davenport proved in 1939 that $G(4) = 16$. But apart from this no other exact values of $G(k)$ are known. It is conjectured that $G(k) = 2k + 1$ if k is not a power of 2 and $G(2^m) = 2^{m+2}$ for $m > 1$. First estimate for $G(k)$

Box 1.

Edward Waring (1736-1798) was the son of John Waring, a farmer in Shropshire, England. He went to Shrewsbury school and entered Magdalene college, Cambridge in 1753. There he impressed his teachers with his mathematical abilities and he graduated BA in 1757 as a senior wrangler. Waring's most famous work, *Meditationes Algebraicae*, was composed in 1759 when he was in Magdalene college and he submitted it to the Royal Society. Though it formed the basis on which he was selected to the Lucasian Chair of Mathematics in Cambridge (once held by Newton) in 1760 at the young age of 24, it was not published untill 1770. He wrote a few other books on number theory and geometry but his writing was not lucid and so did not attract the attention it deserved. In fact, many of his results were rediscoverd by others and go by their names for example, Wilson's theorem. He was elected to the Royal Society in 1763 and was awarded the Copley Medal in 1784. Thomas Thomson's assessment of Waring is perhaps the most accurate one describing him: *Waring was one of the profoundest mathematicians of the eighteenth century; but the inelegance and obscurity of his writings prevented him from obtaining that reputation to which he was entitled.*

Hardy and Littlewood was $G(k) \leq (k-2)2^{k-1}+5$. Substantial contributions, mainly by I M Vinogradov, H M Davenport, R C Vaughan and K Thanigasalam culminated in the work of T D Wooley in 1992 who proved that

$$G(k) \leq k(\log k + \log\log k + O(1)).$$

This remains to be the best result so far.

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Errata

Resonance, Vol.8, No.10, October 2003.	*Resonance*, Vol.9, No.4, April 2004.
Page 55: The Sanskrit sloka should read as उदये सविता रक्तः, रक्तश्चास्तमने तथा। सम्पत्तौ च विपत्तौ च महताम् एकरूपता।।	Page 37: The structure of geraniol should be CH_2OH (±) geraniol