From Lintearia to Lemniscate II: Gauss and Landen's Work

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In the first part [1], we introduced and discussed the '*elastic curve*' and the related algebraic curve, the '*Lemniscate*'. The name 'elastic curve' was given due to the fact that it is the shape assumed by a uniform elastic rod when bent in to a plane curve under a stress of a certain kind. The lemniscate is an algebraic curve (i.e., it is given by polynomial equations) unlike the elastic curve, but the arc lengths are the same for both. Recall that the equation of the lemniscate is

$$(x^2 + y^2)^2 = x^2 - y^2$$

and that it has a parametrisation

$$x^2 = \frac{r^2 + r^4}{2}, \quad y^2 = \frac{r^2 - r^4}{2}.$$

Using this, (or the polar equation $r^2 = \cos 2\theta$) the element of arc length ds is given by $ds^2 (= dr^2 + r^2 d\theta^2) = \frac{dr^2}{1-r^4}$.

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As we saw, these curves gave birth to the theory of elliptic functions. The work of Fagnano on doubling the arc length of the lemniscate and the 'addition theorem' of elliptic integrals due to Euler were discussed. Euler's result includes the fact that if two lemniscate arcs are given by their end points, they can be added with the aid of ruler and compass. Euler began his researches on elliptic integrals (beginning with the lemniscate integral $\int_0^x \frac{dx}{1-x^4}$) with his addition theorem. Let us recall this; Euler took

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$$r = \frac{u\sqrt{1-v^4} + v\sqrt{1-u^4}}{1 + u^2 v^2}$$

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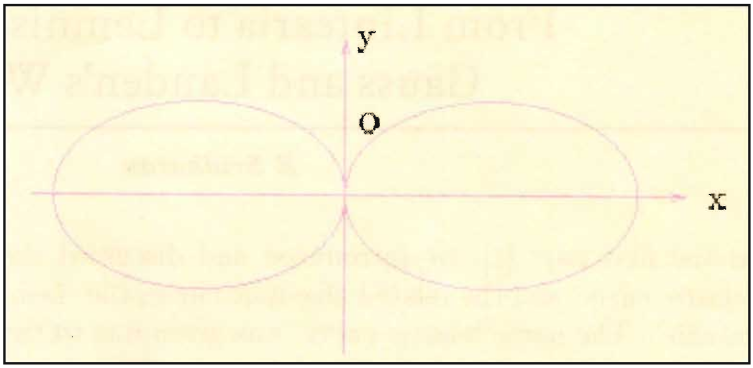
Figure 1.



and he proved

$$\int_0^u \frac{du}{\sqrt{1-u^4}} + \int_0^v \frac{dv}{\sqrt{1-v^4}} = \int_0^r \frac{dr}{\sqrt{1-r^4}},$$

where

$$r = \frac{u\sqrt{1-v^4}+v\sqrt{1-u^4}}{1+u^2v^2}. \qquad (*)$$

Let us pause for a moment and note that this result can be interpreted in the following way. (Actually, Euler missed this interpretation.)

Let us write

$$s(x) = \int_0^x \frac{dx}{\sqrt{1-x^4}} \qquad 0 \le x \le 1.$$

Then the function $s(x)$ is the analogue of the $\sin^{-1}$ function defined by

$$\sin^{-1} x = \int_0^x \frac{dx}{\sqrt{1-x^2}}$$

and the above result of Euler is really an 'addition theorem' for the inverse function l defined by

The addition theorem of Euler is really the addition of the 'lemniscate function'.

$$x = l(s(x)).$$

The inverse function l of s exists, since the function s is monotonically increasing in [0,1]. Then the addition theorem of Euler is really the addition of the 'lemniscate function' l. This discovery of the lemniscate function l had however to wait for Gauss, Abel and Jacobi.

Let us point to one more beautiful discovery of Euler in our context. Let

$$v = \frac{u\sqrt{1-c^4} + c\sqrt{1-u^4}}{1+c^2u^2},$$

where c is a constant u and v are variables. We rationalize this equation to get

$$(v(1+c^2u^2) - u\sqrt{1-c^4})^2 = (c\sqrt{1-u^4})^2.$$

That is

$$v^2(1+c^2u^2)^2 + u^2(1-c^4) - 2uv(1+c^2u^2)\sqrt{1-c^4} = c^2(1-u^4).$$

We therefore have

$$v^2+2c^2u^2v^2+c^4v^2u^4+u^2-u^2c^4-2uv(1+c^2u^2)\sqrt{1-c^4} = c^2(1-u^4)$$

which leads to

$$c^2u^2v^2+u^2+v^2-2uv\sqrt{1-c^4}-c^2+c^2u^2(c^2u^2v^2+u^2+v^2- 2uv\sqrt{1-c^4}-c^2) = 0$$

i.e. $(1+c^2u^2)(c^2u^2v^2+u^2+v^2-2uv\sqrt{1-c^4}-c^2) = 0$.

Note that $1+c^2u^2 \neq 0$ (since we are over real numbers), so that we get $c^2u^2v^2+u^2+v^2-2uv\sqrt{1-c^4}-c^2 = 0$. This was called by Euler the 'complete integral' of the differential equation

$$\frac{du}{\sqrt{1-u^4}} + \frac{dv}{\sqrt{1-v^4}} = 0.$$

Indeed, $c^2u^2v^2 + u^2 + v^2 - 2uv\sqrt{1-c^4} - c^2 = 0$ is a nontrivial solution of the differential equation–the 'general solution' with a constant c. In particular, if we set $c = 1$, we get

$$u^2v^2 + u^2 + v^2 - 1 = 0$$

which is typically the equation of an 'elliptic curve' in the spirit of Euler.

Led by his work on 'integration' of the lemniscate differential equation above, Euler began considering systematically more general differential equations of the type,

$$\pm\frac{dx}{\sqrt{F(x)}} = \pm\frac{dy}{\sqrt{G(y)}},$$

where F and G are polynomials in x and y of degrees 3 or 4. In fact these arise out of the following situation [4]: Consider a polynomial equation $\phi(x, y) = 0$ of degree ≤ 2 in x and y with coefficients in a subfield of real numbers. Such an equation can be written as

$$\phi(x, y) = P_0(x)y^2 + 2P_1(x)y + P_2(x) = 0$$

and also as

$$\phi(x, y) = Q_0(y)x^2 + 2Q_1(y)x + Q_2(y) = 0,$$

where P_i, Q_i are polynomials in x and y respectively of degrees ≤ 2. By solving either for y or x, we see that the curve defined by $\phi = 0$ is 'birationally equivalent' to either of the curves

$$z^2 = F(x) \quad \text{and} \quad w^2 = G(y),$$

where $F = P_1^2 - P_0P_2$, $G = Q_1^2 - Q_0Q_2$. The relation $\phi = 0$ serves as an 'isomorphism' between the curves $z^2 = F(x)$ and $w^2 = G(y)$.

For example, if $\phi : x^2y^2 + x^2 + y^2 - 1 = 0$, we get $F = 1 - x^4$, $G = 1 - y^4$ and we obtain the curves

$z^2 = 1 - x^4$ and $w^2 = 1 - y^4$ which are birationally equivalent to the curve $x^2y^2 + x^2 + y^2 - 1 = 0$.

Returning to the general case, we have

$$\frac{1}{2}\frac{\partial\phi}{\partial x} = Q_0(y)x + Q_1(y) = \pm\sqrt{G(y)}$$

$$\frac{1}{2}\frac{\partial\phi}{\partial y} = P_0(x)y + P_1(x) = \pm\sqrt{F(x)}$$

so that $0 = d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$ gives the differential equation

$$\pm\frac{dx}{\sqrt{F(x)}} = \pm\frac{dy}{\sqrt{G(y)}}$$

Thus the isomorphism ϕ between $z^2 = F(x)$ and $w^2 = G(y)$ 'transforms' the differential $\frac{dx}{z}$ to $\frac{dy}{w}$. Euler used for instance this 'canonical equation' $\phi = 0$ in solving 'diophantine' problems, namely that of finding solutions of $\phi(x, y) = 0$ starting with one, by his 'ascent method'. We describe this method briefly.

In fact, if (x_1, y_1) is a solution of $\phi(x, y) = 0$, we get that

$$\phi(x, y_1) = Q_0(y_1)x^2 + 2Q_1(y_1)x + Q_2(y_1) = 0$$

has a solution to be $x = x_1$. But, in general, a quadratic equation in x must have two solutions: we choose the other solution $x = x_2 = \frac{-2Q_1(y_1)}{Q_0(y_1)} - x_1$. Since

$$\phi(x, y_1) = P_0(x)y_1^2 + 2P_1(x)y_1 + P_2(x),$$

$\phi(x_2, y)$ has solution $y = y_1$. But then it must have another solution given by $y_2 = \frac{-2P_1(x_2)}{P_0(x_2)} - y_1$, so that $\phi(x_2, y_2) = 0$. So, starting with the solution (x_1, y_1) we end up with another solution (x_2, y_2). In general, this procedure would go on to yield infinitely many solutions of $\phi = 0$. (On the other hand, this procedure could also stop and the solution could turn back.)

The lemniscate has always been considered as an analogue of the circle and $\tilde{\omega}$ should then be thought of as an analogue of π.

We finally discuss some great work of Gauss on the lemniscate (also that of Abel and Jacobi). We begin by noting that the arc length $s = s(x)$ of the lemniscate in the first quadrant is given by

$$s(x) = \int_0^x \frac{dx}{\sqrt{1-x^4}} \quad 0 \le x \le 1.$$

In particular, the total arc length of the lemniscate is $2\tilde{\omega}$, where

$$\frac{\tilde{\omega}}{2} = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

The lemniscate has always been considered as an analogue of the circle and $\tilde{\omega}$ should then be thought of as an analogue of π. One of the beautiful results of Gauss was the computation $\tilde{\omega}$. First of all, let us note that if we put $x = \cos\theta$, the integral transforms into

$$\int_{\pi/2}^0 \frac{-\sin\theta d\theta}{\sqrt{1-cos^4\theta}} = \int_0^{\pi/2} \frac{\sin\theta d\theta}{\sin\theta\sqrt{1+\cos^2\theta}}$$
$$= \int_0^{\pi/2} \frac{d\theta}{\sqrt{2\cos^2\theta+\sin^2\theta}}.$$

Gauss set out to compute (more generally) integrals of the form

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}},$$

where a and b are real numbers with $a \ge b > 0$. (Note that if $a = b$, then the integral is trivially seen to be $\frac{\pi}{2a}$.) Gauss reduced the computation in the general case to this trivial case. In order to achieve this, he used a new concept, the notion of 'arithmetic geometric mean' of two real numbers a and b with the condition $a \ge b > 0$. Define inductively

$$a_1 = \frac{a+b}{2}, \quad a_n = \frac{a_{n-1}+b_{n-1}}{2} \quad n \ge 2$$

$$b_1 = \sqrt{ab}, \quad b_n = \sqrt{a_{n-1}b_{n-1}}$$

We obviously have

$$a \geq a_1 \geq a_2 \geq \quad \geq a_n \geq b_n \geq b_{n-1} \geq \quad \geq b_1 \geq b$$

for every n. Note also that $0 \leq a_n - b_n \leq \frac{1}{2^n}(a-b)$ for every n. This implies that $\{a_n\}$ and $\{b_n\}$ are both convergent sequences and that they converge to the same limit. This limit is called the *arithmetic geometric mean* of a, b and denoted by $M(a,b)$. Note that $M(a,a) = a$. Gauss proved the following (see also [5]):

Landen was the first to introduce what we now call a 2-isogeny of an elliptic curve.

Theorem:

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \frac{\pi}{2M(a,b)}.$$

From this, we have the

Corollary : $$\frac{\widetilde{\omega}}{2} = \frac{\pi}{2M(\sqrt{2},1)}.$$

Before proving the theorem, we make a few remarks.

Landen (1710-1790) who was a British mathematician, was the first to introduce what we now call a 2-isogeny of of an elliptic curve which he applied to transform the integral in question, using which, one can also compute the integral. Landen is still remembered for his work and there is a Landen's point related to the motion of a pendulum!

Gauss introduced an ingenious substitution to prove the theorem above.

What we shall give here is a slightly different substitution due to D J Newmann [2] which is equally clever.

We first note that if we substitute $t = b\tan\theta$, we have

$$2\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} =$$

$$2\int_0^\infty \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}.$$

Now with $a_1 = \frac{a+b}{2}$, $b_1 = \sqrt{ab}$ and for $0 < t < \infty$, let $x = \frac{1}{2}(t - \frac{b_1^2}{t})$.

Then

$$\sqrt{(t^2+a^2)(t^2+b^2)} = \sqrt{t^4 + t^2(a^2+b^2) + a^2b^2} =$$

$$\sqrt{t^4 + t^2(4a_1^2 - 2b_1^2) + b_1^4}$$

$$= \sqrt{(t^2 - b_1^2)^2 + 4t^2a_1^2} = \sqrt{t^2(t - \frac{b_1^2}{t})^2 + 4t^2a_1^2}$$

$$= \sqrt{4t^2x^2 + 4t^2a_1^2} = 2t\sqrt{x^2 + a_1^2}$$

Also, $t^2 + b_1^2 = t(t + \frac{b_1^2}{t}) = t\sqrt{(t + \frac{b_1^2}{t})^2} = t\sqrt{4x^2 + 4b_1^2} = 2t\sqrt{x^2 + b_1^2}$. Note that $0 < t < \infty$ implies $-\infty < x < \infty$. We also have $2t^2 dx = (t^2 + b_1^2)dt$. Hence

$$\int_0^\infty \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{\sqrt{(x^2+a_1^2)(x^2+b_1^2)}}$$
$$= \int_0^\infty \frac{dx}{\sqrt{(x^2+a_1^2)(x^2+b_1^2)}}.$$

Whatever we did for the pair (a, b), we do for (a_1, b_1), so that we get eventually for all $n \geq 1$,

$$\int_0^\infty \frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}} = \int_0^\infty \frac{dt}{\sqrt{(t^2+a_n^2)(t^2+b_n^2)}}.$$

Now taking limits

$$\lim_{n\to\infty}\int_0^\infty \frac{dt}{\sqrt{(t^2+a_n^2)(t^2+b_n^2)}} = \int_0^\infty \frac{dt}{\sqrt{(t^2+M(a,b)^2)^2}},$$

since we are obviously permitted to take the limit inside; we have then that

$$\int_0^\infty \frac{dt}{(t^2+a^2)(t^2+b^2)} = \int_0^\infty \frac{dt}{t^2+M(a,b)^2} = \frac{\pi}{2M(a,b)}.$$

Gauss defined the notion of the arithmetic geometric mean also for a pair of complex numbers and in this case the definition is more subtle. Gauss uses this notion to study 'modular functions'.

One should remark that Gauss defined the notion of the arithmetic geometric mean also for a *pair of complex numbers* and in this case the definition is more subtle. Gauss uses this notion to study 'modular functions' We shall not however go into this deep area of mathematics.

We end by singling out just one more great triumph of Gauss related to the lemniscate. As is well known, Gauss solved, when he was only nineteen years old, the very ancient problem (left open by the Greeks) of determining those integers $m \geq 1$ for which a circle can be divided into m equal parts, by using ruler and compass. The Greeks could construct the regular pentagon with ruler and compass (which is equivalent to the construction of the so called 'golden ratio' which was a favourite of the Greeks). But the question of deciding, for which prime p bigger than 5, can regular polygons of p sides be inscribed in a circle by ruler and compass, was open for several centuries. Gauss completely solved this problem by showing that the only integers m for which a regular polygon of m sides can be inscribed in a circle with ruler and compass are those of the form $m = 2^a \; p_1 p_2 \quad p_r$, where a is any integer ≥ 0 and p_i are distinct Fermat primes, i.e. distinct primes of the form $2^{2^k} + 1$ (k, an integer, ≥ 0). In particular, Gauss explicitly constructed the 17 sided regular polygon. As A Weil remarks [3], Gauss certainly merited the 'Fields Medal' for his achievement! Gauss described his proof in his famous book *Disquisitiones* and added that he had the same result for the lemniscate curve. He however never published a proof. To put the record straight, Gauss did have a proof that the lemniscate can be divided into five equal parts in his famous diary which was published only several years after his death. (As we have noted earlier, Fagnano had already solved the problem of halving the lemniscate curve). Abel, who read Gauss' *Disquisitiones*, was intrigued by Gauss' remark, worked on it and published a proof of the analogous theorem

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Note:
Part I [1], page 27, on line 6 from the bottom: $v = \cos x$ should read $v = \sin y$.

for the lemniscate. In fact, this beautiful work of Abel depended on his theory of lemniscate functions, which as we have said earlier, are indeed examples of elliptic functions in modern terminology. Essentially at the same time, Jacobi also studied elliptic functions in great depth.

To sum up, the elastic curve was a blessed object which has been responsible for the enrichment of mathematics.

Suggested Reading

[1] R Sridharan, From Lintearia to Lemniscate I, *Resonance*, Vol.9, No.4, pp.21-29, 2004.

[2] D J Newman, A simplified version of the fast algorithms of Brent and Salamin, *Math. Comp.*, Vol.44, pp.207-210, 1985.

[3] A Weil, La cyclotomie jadis et naguère, *L'Enseignement mathém*, Vol. XX, pp.247-263, 1974.

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[5] B Sury, The arithmetico-geometric mean of Gauss, *Resonance*, Vol. 5, No. 8, pp.71-83, 2000.

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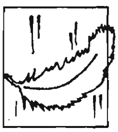

"I hope that I have at least left you with the impression that mathematics is an extremely complex creation, which exhibits so many essentially common traits from Art and from both the experimental and theoretical sciences. It reflects simultaneously all three of them and therefore must be distinguished from all three of them."

– *A Borel*
in a lecture to the
Siemens Foundation
on May 7, 1981.