Ramanujan: Essays and Surveys

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Ramanujan: Essays and Surveys
Bruce C Berndt and Robert A Rankin
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The fact that Ramanujan made path-breaking discoveries without much formal training and was a misfit under the regular curriculum (he failed in the FA examination), has been a source of mystery to many. This intriguing aspect has fired the imaginations of several authors who have penned his biography. Almost all of them have dealt more with his personal life than with his mathematical work. This volume can be said with some justification to be an attempt at a more mathematical biography. A facsimile edition of Ramanujan's famous notebooks had been published in 1957 by the Tata Institute of Fundamental Research, Bombay and, as such, they were unedited.

One of the two editors, Robert Rankin, passed away just before this volume was published. The other editor, Bruce Berndt, has devoted, since May 1977, all of his research efforts to editing Ramanujan's notebooks. No amount of praise is adequate acknowledgement of his contribution to the mathematical community in this regard.

This volume has some rather unusual contents too which we shall describe before moving on to its mathematical contents. First, the only four extant photographs of Ramanujan are reproduced and their backgrounds discussed. Then, there is an interesting account by a British physician D B A Young on Ramanujan's illness and its (faulty) diagnosis. It has been widely believed that tuberculosis was the cause of Ramanujan's death but the doctor explains in detail how and why the diagnosis was faulty and opines that hepatitic amoebiasis was the most likely cause of his illness. Apparently, inadequately treated amoebiasis can be a permanent infection even though the patient could spend long periods without obvious symptoms. A brief biography of Janaki Ammal (Ramanujan's widow) by Bruce Berndt appears in this volume and is followed by an interview with her by P Nandy. There is also a biography of Narayana Aiyar, the Chief Accountant of Madras Port Trust and Ramanujan's close friend, who supported and encouraged him while in Madras. In fact, Ramanujan's first letters to Hardy were written with the help of Narayana Aiyar.

A highly interesting discussion is that of the books studied by Ramanujan in India. It is well known that Ramanujan studied G S Carr's *A synopsis of elementary results in pure mathematics* and adapted its style in recording his results. However, one of the veritably intriguing questions that arose was how Ramanujan was able to prove hundreds of identities in the theory of modular elliptic functions without learning complex analysis in a conventional manner. The editors unearth the fact that in 1914, the library of the University of Madras contained a copy of the book, *The applications of elliptic functions* by A G Greenhill. This book also does not use any complex analysis. They go on to point out other features which seem to indicate that Ramanujan must have read this book and was influenced by it in his work on modular equations.

An extremely fascinating overview of Ramanujan's three notebooks is given by Berndt. He counts some 3254 results recorded in them. But, as is well known, as much as half of these may have been rediscovery. As a possible reason for Ramanujan recording only the final result, and not the proofs, Berndt says that like most Indian students of those times, Ramanujan wrote on a slate and rubbed out proofs; paper was expensive. Berndt emphasizes that Ramanujan thought like other mathematicians and not through divine intervention or anything of that sort. He just thought with more insight than most people. Berndt arranges the contents of this chapter into 12 basic topics and discusses each in some detail. For instance, people who think of Ramanujan primarily as a number-theorist may be surprised to learn that several hundred theorems of his on theta functions and modular functions, are at the interface of analysis and number theory. Berndt says that infinite series were Ramanujan's first love, and his talents in working with them were possessed perhaps only by the great Euler.

George Andrews who discovered the so-called Ramanujan's 'lost' notebook, had written a beautiful account of this discovery as well as of the contents of the 'lost' notebook in the *American Mathematical Monthly*, 1978. This is reproduced in this volume. In 1976, while sorting through some materials from the estate of G N Watson, Andrews found a manuscript of about 100 pages in Ramanujan's handwriting. Andrews contends that these were written during the last year of Ramanujan's life, after returning to India, when he was in 'severe pain and only skin and bones.' Of the contents, only some results on 'mock theta functions' came to be known earlier through the last letter that Ramanujan wrote to Hardy. Perhaps Watson filed this lost notebook away and forgot about it. Watson gave a lecture in 1935 on some of Ramanujan's results on the mock theta functions quoted in the last letter mentioned above. The title of the lecture was, 'The final problem: an account of the mock theta functions.' He says, "I doubt whether a more suitable title could be found for it than the title used by John H Watson, MD, for what he imagined to be the final memoir on Sherlock Holmes." Concluding his address, Watson says, "Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance."

Another captivating article in the volume describes 58 problems that Ramanujan had posed in the *Journal of the Indian Mathematical Society*. Although these appear also in Ramanujan's collected works, here they are not merely reproduced but are analysed in some detail. Following this, Freeman Dyson's article, 'A walk through Ramanujan's garden', published immediately after Ramanujan's birth centenary, is reproduced. The following samplings from his article bear witness to his feelings on Ramanujan's mathematics:

"My love affair with Ramanujan's mathematics began 48 years ago. The wonderful thing about Ramanujan was that he discovered so much, and yet left so much for other people to discover. I have intermittently been coming back to Ramanujan's garden and, every time, I find fresh flowers blooming."

"In the cold dark evenings, while I was scribbling these beautiful identities amid the death and destruction of 1944, I felt close to Ramanujan. He had been scribbling even more beautiful identities amid the death and destruction of 1917."

Ramanujan's vast work on hypergeometric series was surveyed by Richard Askey in an article published in 1988. This is also reproduced here. Finally, I come to my two favourite articles in this volume. One is by J M Borwein and P B Borwein on 'Ramanujan and pi.' Ramanujan wrote a paper, 'Modular equations and approximations to π where one of his formulae reads

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}.$$

The Borweins assert that the partial sums in the above infinite series converge to the true value more rapidly than any other calculation of π until the 1970's. Each successive term adds roughly eight more correct digits. The Borweins bettered Ramanujan's result in 1987. In an article in Scientific American of 1988, they say, "Iterative algorithms (where the output of one cycle is taken as the input for the next) which rapidly converge to pi were, in many respects, anticipated by Ramanujan, although he knew nothing of computer programming. Indeed, computers not only have made it possible to apply Ramanujan's work but have also helped to unravel it. Sophisticated algebraic manipulations software has allowed further exploration of the road Ramanujan travelled alone and unaided 75 years ago."

A sense of incredulity prevails on reading these words when one pictures Ramanujan sitting and writing on a slate and erasing with his sleeve!

Finally, there is the extremely educative and fascinating article by Atle Selberg on Ramanujan's work and how his own work was influenced by it. Selberg says, "Ramanujan's particular talent will seem to be primarily of an algebraic and combinatorial nature. He developed it, for a long time in complete isolation, really without any contact with other mathematicians. He had, on his own, acquired an extraordinary skill of manipulation of algorithms, series, continued fractions and so forth, which certainly is completely unequalled in modern times."

"He might very well have become a theory-builder if he had had a different and more conventional start and training as a mathematician. Even then, in what has been left in his work, there seems quite clear evidence that he had developed, on his own, a theory of modular forms and equations."

At another place, Selberg recounts some intriguing facts on Ramanujan's work on partitions. For instance, he points out that in his very first letter, Ramanujan has written an exact formula for an infinite product analogous to that defining the function $p(n)$ of partitions of n; he had suggested also the existence of such a formula for $p(n)$ itself. Selberg opines that perhaps Hardy did not fully trust Ramanujan's insight and chose an expression for $p(n)$ which gives only an asymptotic formula. Perhaps, Ramanujan did not press this point out of respect for Hardy. Later, Rademacher came out with an exact formula for $p(n)$, whose similarity with Ramanujan's exact formula mentioned above is so striking that it is surprising that Hardy and Ramanujan did not end up with it themselves.

Selberg speculates about what would have happened had Ramanujan come into contact with Hecke, a great mathematician of talents more similar to Ramanujan's. he says that that might perhaps have brought out new things in Ramanujan which did not come into fruition by his contact with Hardy. But, he adds that Hardy deserves the greatest credit for recognizing Ramanujan's originality and assisting him and his work in the best way he could.

In December 2003, there was a meeting of the American Mathematical Society in Bangalore and, George Andrews gave a talk on 'Ramanujan and partial fractions.' He said that after several years trying to recognise and understand the specific insights that Ramanujan had, he feels now that a lot of Ramanujan's identities are based on his highly developed intuition to recognise partial fraction expansions in diverse situations.

Ramanujan's work on cusp forms and their generalisations play a central role in present-day mathematics and other deep discoveries like the mock theta functions are still in the early stages of being understood. Thus, Ramanujan's stature in mathematics has only been growing over the years and this volume will be a valuable source of initiation into Ramanujan's mathematics for some time to come. To sum up:

Ramanujan did mathematics somehow;
we still can't figure out even now.
He left his mark on 'p of n',
wrote π in series quite often.
The theta functions he called 'mock'
are subject-matter of many a talk.
He died very young – yes, he too!
He was only thirty-two!
His name prefixes the function τ.
Truly, that was his last bow!

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