

The No-Slip Boundary Condition in Fluid Mechanics

2. Solution of the Sticky Problem

Sandeep Prabhakara and M D Deshpande

Ideas leading to the resolution of the problem of no-slip condition for fluid velocity at a solid surface are traced in this concluding part of the article. In the continuum limit velocity slip being zero is established beyond any doubt now. Even turbulent flows which have a large velocity gradient near a wall have to satisfy the no-slip condition at every instant. From molecular considerations, on the other hand, we know that the velocity slip is proportional to the mean free path which may not be negligible in rarefied gas flows. Experimental verification of no-slip has been only indirect and it is only recently that slip velocity in non-wettable liquids has been measured directly.

1. Historical Development

A brief and excellent review of this problem of velocity slip in fluid flow is given in the book by Goldstein [1]. We freely borrow from this book adding some explanations and supplements based on the earlier discussion in Part 1.

We saw that Newton tacitly assumed the no-slip condition in the analysis of vortex motion but he missed it in the problem of the cylinder moving along its length. Daniel Bernoulli recognized as early as in 1738 that a fluid could not slip freely over a solid surface. The well known Bernoulli equation giving a relation between pressure and velocity in a fluid is valid only for an inviscid or frictionless fluid. Based on the discrepancies between the measured data for a real fluid and the calculated data for an ideal fluid he concluded that perfect slip was not possible. But it is going only half way; it does not mean no-slip was meant. Based on the observation of water flow in a channel Du Buat concluded in 1786 that the fluid adjacent to the surface was at



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rest, but with a qualification that this is subject to the condition that flow velocity in the channel is *sufficiently* small. This is a brave conclusion in spite of the cautious qualification. It is quite possible he was influenced by the model of a rolling ball (see *Box 1* in Part 1) which may roll without slipping at low velocities but may slip at higher velocities.

Coulomb addressed this problem also and his experiments were brilliant and were a logical extension of his experiments on dry friction. He took a metallic disk oscillating in water and smeared it with grease and later covered the grease with powdered sandstone. To his surprise the resistance of the disk scarcely changed in either case. This is a remarkable result. It appears strange initially since our intuition is based mainly on friction between solid surfaces. We know that if we grease them the friction decreases. Polishing the surfaces also leads to a decrease in friction. Coulomb might have expected that a greased surface leads to better slip. The result Coulomb arrived at is surprising but is similar to what we have seen for the Hagen–Poiseuille flow. We saw that the resistance coefficient $\lambda = 64/Re$ and it depended only on Reynolds number Re but not on the surface conditions. (*Figures 1* and 2). This is not any less surprising. We may add, however, that λ does depend on the surface roughness (higher λ for rougher surfaces) for turbulent flows.

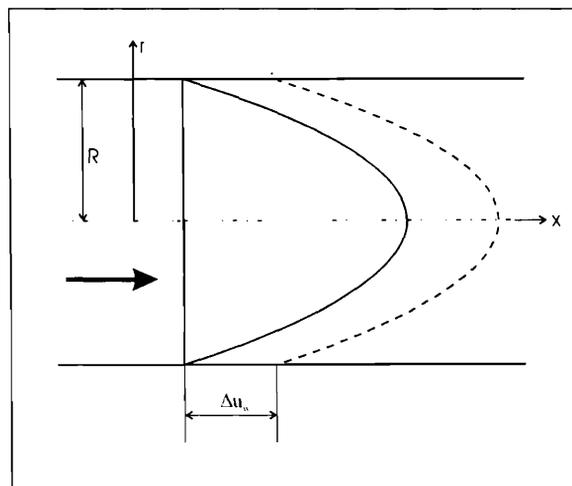


Figure 1. Parabolic velocity profile in a fully developed pipe flow with and without slip.

Notice that conclusions mentioned above and arrived at during the eighteenth century by Bernoulli, Du Buat and Coulomb came from experimental observations and before the N-S equations were known. During the nineteenth century three different hypotheses were put forward by various authors at different times. They will be discussed in the next section.

2. The Struggle to No-Slip

The first of the three hypotheses we are going to discuss assumes that velocity of the fluid at the wall is the same as that of the moving surface itself and it changes continuously inside the fluid. This is the no-slip condition and it seems to have been Coulomb's belief. The second one was put forward during the second decade of the nineteenth century by Girard who did experiments on the flow of liquids through tubes. He supposed that a very thin layer of fluid remains attached to the walls and the bulk of the fluid slips over the outer surface of this stagnant layer. Further, he supposed that if the wall (fluid) material remains the same the thickness of the stagnant layer is constant. This means that this layer presents to the moving fluid the same irregularities as those of the wall itself. Because of the choice of such a model he was obliged to make other assumptions. For a liquid such as mercury that does not wet the glass tube wall, he supposed that the thickness of the layer was zero and the liquid slips over the surface. It is not too surprising how the ideas of wettability and no-slip got mixed up since even though they are distinct concepts they are closely connected. Wettability is related to surface tension and contact angle α , ($\alpha > 90^\circ$ is termed as non-wettable) and comes into picture only when there is a solid-liquid-gas contact line leading to a free surface. No-slip, on the other hand, does not need a free surface. Effect of a non-wettable surface on slip is discussed in *Box 1*. Note that the presence of a stagnant layer with slip leads to a discontinuity in velocity in the fluid. Now we know that discontinuities cannot exist in a real (or viscous) fluid since it

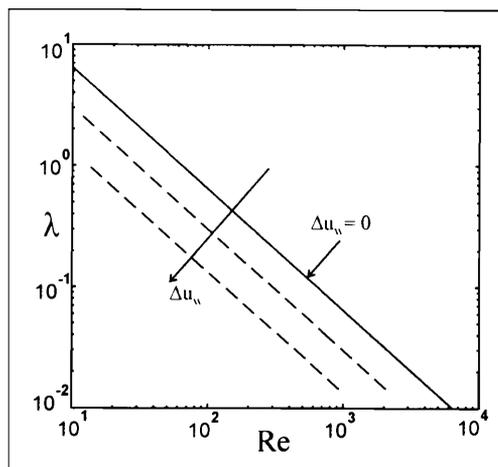


Figure 2. Variation of resistance coefficient λ as a function of Re for a fully developed pipe flow with and without slip.

During the nineteenth century three different hypotheses regarding fluid slip at the wall were put forward. These competing ideas had their own supporters.

Box 1. Non-wettability and Slip

In gases the molecules are required to remain chemically adsorbed onto a solid surface for a sufficiently long time to force no-slip. See also Section 4. In liquids also it is generally true except when the liquid does not wet the surface.

Watanabe, Udagawa and Udagawa [7] have made measurements in tubes, as large as 12mm in diameter, and with specially coated highly water-repellent surfaces, with contact angle $\alpha \approx 150^\circ$. Their direct measurements show slip and drag reduction of about 14% and their results resemble *Figures 1 and 2*. They have directly verified the Navier's equation (1). Drag reduction was present in the laminar and transitional regimes but not for the turbulent case. They also propose that for slip to occur α required should be at least 120° . For water on a smooth teflon surface $\alpha < 110^\circ$ and hence there is no slip. Note that for Mercury-air-glass $\alpha \approx 130^\circ$.

Similar experimental verification of slip is also done by Tretheway and Meinhart [5] in micro-channels with a specially prepared water-repellent surface. Measurements of Brutin and Tadrist [3] in micro-tubes, on the other hand, show an increased drag due to the ionic composition of water. Drag value was brought back to the classical value of $64/Re$ by using deionized water (distilled water) and also by deactivating the surface.

leads to infinite stress. But this model was proposed before the N-S equations were known.

The third hypothesis was due to Navier himself. From the molecular hypothesis which led him to the correct equations of motion he deduced in 1823 that there is (partial) slipping at a solid boundary. He argued that the wall resists this slipping with a force proportional to the slip velocity u_s . Since this tangential stress has to be continuous from the solid wall to the fluid he assumed for flow in one direction

$$\beta u_s = \mu \partial u / \partial n \quad (1)$$

where n is along the normal to the wall and β is a constant with μ/β being a length. This length is zero if there is no slip. Navier explained Girard's experimental results for flow through tubes using this model. Note that there is no velocity discontinuity inside the fluid in this model.

It is interesting to note here that Poisson obtained similar conditions as Navier's but suggested that these should be ap-

Navier deduced from molecular hypotheses that there is partial slip at the wall, the wall resistance being proportional to the slip velocity itself.



plied at the outside of a stagnant layer. Stokes, another giant and who independently derived the equations of motion, was initially inclined towards the first (i.e. the correct no-slip) hypothesis but then wavered between this hypothesis and Navier's. It was because his calculations did not agree with the experimental data for pipe flow known to him. His calculations were correct and they would have agreed with the experimental results of either Hagen or Poiseuille. In his report to the British Association in 1846 he mentioned all three hypotheses without picking any one. But finally he decided on the first (i.e. no-slip) based on two arguments

- (i) Existence of slip would imply that the friction between a solid and fluid was of a different nature from, and infinitely less than, the friction between two layers of fluid.
- (ii) Satisfactory agreement between the results obtained with no-slip assumption and the observations.

The first argument here is remarkable. A tangential stress inside a fluid leads to a deformation of the fluid element but still the velocity is continuous (as is known from the observations). Why should it not be true, one may ask, at the solid-fluid interface also? For a given stress a larger deformation results if the viscosity is small. If we get a discontinuity in velocity at the interface that means the mechanism of friction between the solid and the fluid should be different and also it should be infinitely less than between two layers of fluids. Looking back it was a convincing argument from Stokes. But the no-slip condition seems to have appeared unnatural and the competing ideas had their own supporters. We have seen how Hagen and Poiseuille did the experiments but did not zero in on the no-slip condition. Even Darcy (1858) and Helmholtz (1860) settled for some form of slip!

By the end of the nineteenth century the hypothesis supported by Stokes was accepted. Finally it had to be, of course, because it is true. But to achieve that acceptance there were discussions at length. Many experiments were done and repeated. This was

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Whetham did careful experiments to conclude that there was no slip at the wall. He also repeated the experiments of other investigators and removed the doubts that there was support for slip. But we should keep in mind that these verifications are only indirect.

because we have to know if there is a small slip at the wall or is it exactly zero. Experiments on oscillating glass disks in air by Maxwell and several other experiments including flow of mercury in glass tubes were specially designed to investigate the velocity slip. Most of these experiments were concerned with the laminar flow. Noteworthy is the conclusion by Couette in 1890 that even the turbulent flows have to satisfy the no-slip condition, despite a very large gradient near the wall!

The details of the experiments by Whetham will be given in the next section which in a way settled the issue for no-slip.

3. Careful Experiments by Whetham

Whetham [8] did a series of careful experiments to compare the time taken for a given volume of water to discharge through a glass capillary tube. After a set of measurements the capillary tube was removed, its inside silvered to form a thin smooth layer and experiments repeated. Then the silver layer was dissolved off with nitric acid. From the weight of the tube with and without silver coating, the thickness of silver deposited (about $0.014\mu\text{m}$) was estimated. Using Poiseuille solution for the fully developed flow correction was applied for the decrease in tube diameter due to the silver coating and more importantly for change in viscosity due to temperature variation that occurred between two experiments. The change in flow rate due to silver coating, after these corrections were applied, was negligibly small enough to declare that there is no slip at the wall.

It is to the credit of Whetham that he repeated the experiments of Girard (1813-1815) who had measured flow of water through copper tubes and those of Helmholtz and Piotrowski (1860) where measurements of time periods were made for a pendulum formed by a glass bulb filled with different liquids and suspended bifilarly by a fine copper wire. Due to friction at the inside wall of the bulb the Pendulum motion slowed down and its logarithmic decrement was evaluated. The experiment was repeated after the inside of the bulb was silver coated. By



carefully repeating these two experiments Whetham removed the doubts that these experiments had supported a slip. But we should keep in mind that these verifications are only indirect.

It is interesting to note that during this period when still some doubts existed on the basic issue of fluid slip at the wall, a fundamental experiment was done by Osborne Reynolds (1883) to determine when a laminar flow in a pipe became turbulent. Whetham was aware of these results and took care to keep the flow laminar in the tubes he used.

4. Navier, Maxwell and the Molecular Theory

Till now we have assumed that the continuum theory is valid and hence the molecular structure of the fluid is ignored. Then the fluid slip at the solid wall is zero. But what really happens at the molecular level?

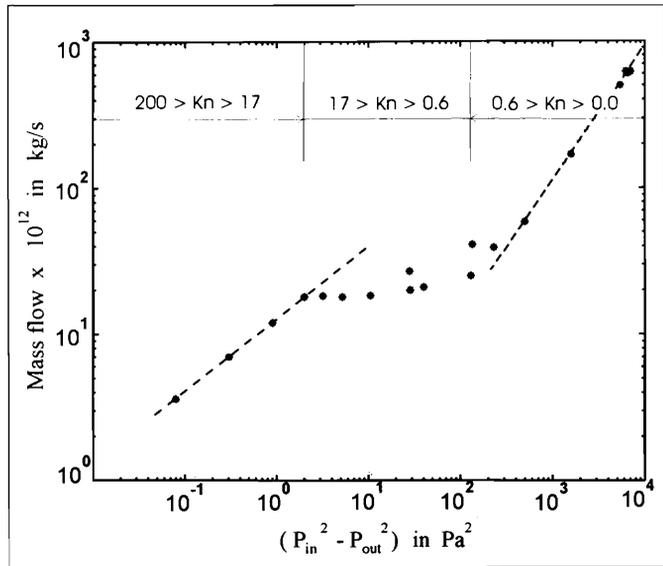
In a gas, characteristic dimension in the dynamics of the molecules is the mean free path. It is the average distance travelled by a molecule between two molecular collisions. Recall that Navier (1823), through a molecular hypothesis had concluded that slipping takes place at the wall and the length scale involved in which it takes place is μ/β . Maxwell (1879) who has done pioneering work in the kinetic theory of gases, concluded that slip takes place according to the equation of Navier and the length μ/β is comparable to Λ , and it may be 2Λ . In the continuum theory Λ is zero. But in a real gas Λ is non-zero but extremely small. In air at normal atmospheric temperature and pressure Λ is $0.065 \mu\text{m}$. In liquids it is still smaller. This non-zero but small value of Λ was where probably one faced the difficulty, both conceptual and practical.

If Λ turns out to be comparable to the characteristic flow dimension L , say pipe radius, then slip at the wall cannot be neglected and also it is easily detected. The ratio Λ/L is the Knudsen number Kn . It is possible to increase Λ (and Kn increases as a result) by decreasing the density of the gas. For $Kn < 0.01$ one

The characteristic dimension in molecular dynamics is the mean free path. If it turns out to be large and comparable to the characteristic flow dimension, say pipe radius, then slip at the wall cannot be neglected.



Figure 3. Mass flow rate for a rarefied gas flow in a 2mm tube of 200mm as a function of $(P_{in}^2 - P_{out}^2)$. Data obtained by S Tison at NIST (Kn is based on P_{out}).



gets the continuum flow and for $Kn > 0.01$ the molecular scales do become important, continuum theory breaks down and slip cannot be ignored. These features are highlighted in *Figure 3*. These results for rarefied gas flow are due to S Tison (1995) and from the book by Karniadakis and Beskok [4]. Here the mass flow rate in a 2 mm diameter tube of 200 mm length is plotted for inlet and outlet pressure variation as shown. Both P_{in} and P_{out} are varied in this experiment and hence it is not possible to replot this graph in the standard form of *Figure 2* (in Part 1). But what is interesting here is that the Knudsen number range is shown and we can clearly identify a shift in the type of flow and also the presence of slip when it occurs. On the right side for $Kn < 0.6$ we have the continuum flow and as we move left and if the pressure square difference falls below 500 Pa², the decrease in mass flow rate is less rapid. This is because of the slip at the wall due to a large mean free path Λ or a large value of Kn . In the free molecular flow regime for $Kn > 17$ the variation in mass flow rate is again linear but with a reduced slope. It is instructive to imagine these flows with large Λ . This figure covers the entire laminar flow regime. High Kn flows with slip are possible in modern engineering applications like MEMS (Micro Electro-Mechanical Systems).

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Box 2. Flow over a Permeable Surface and Associated Slip Velocity

Imagine the channel flow we considered in the section on the Hagen–Poiseuille flow (in Part 1) to be consisting of permeable or porous channel walls as shown in *Figure 4*. The applied pressure gradient induces a flow in the channel and also in the channel walls along x-direction. Even though the no-slip condition is valid on the walls of the individual pores, a slip velocity occurs in the average sense at the interface of the channel wall due to the tangential velocity in the wall. Hence it is convenient to approximate a slip boundary condition rather than consider the flow details inside the porous paths. Interesting experiments have been done by Beavers and Joseph [2] to model such a flow. (See *Figure 4*).

In flow inside a permeable material like sand, the filter velocity is governed by the Darcy’s law

$$u_f = -\frac{k}{\mu} \frac{dp}{dx}, \quad (B2.1)$$

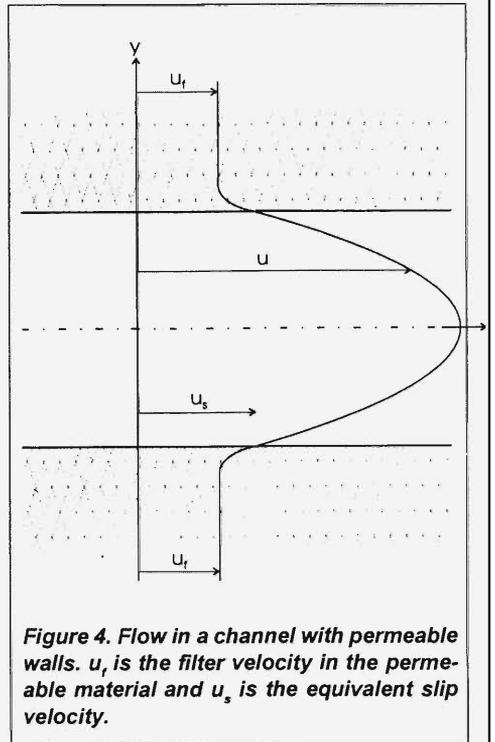
where k is the permeability of the porous material. The true velocity of the fluid satisfies the no-slip condition at the walls of the porous paths and is bound to be higher than u_f in some locations .

Now returning back to the channel flow with a permeable wall, the flow inside the channel can be assumed to have a slip velocity u_s . Inside the channel wall as one moves away from the interface the velocity decreases from u_s to Darcy value u_f given by the equation above. With this picture in mind we would like to solve for the flow in the channel that has permeable walls. The slip velocity u_s is an unknown and it is modelled to be related to the flow inside the channel by

$$\left. \frac{du}{dy} \right|_{y=0} = \frac{\alpha}{\sqrt{k}} (u_s - u_f) \quad (B2.2)$$

where α is a dimensionless constant depending on the material parameters of the permeable wall. The solution leads to a flow rate higher than the usual fully developed case. In other words the resistance coefficient $\lambda = 64 / Re$ we came across in the section on Hagen–Poiseuille Flow (in Part 1) will be modified as indicated by equation (12). Here $\Delta u_w (=u_s)$ depends on α and k as described in Beavers and Joseph [2].

It is interesting to compare the expressions for the slip velocity given by equation (1) due to Navier and equation (B2.2) for the permeable wall with $u_f = 0$. Both the expressions model slip velocity to be proportional to the local velocity gradient or the shear stress.



Experimental justification of no-slip has been indirect and, at best, is only a verification. But we can rely on similitude arguments to justify the N-S equations and the no-slip condition.

5. Confidence in No-Slip

We have seen that in the continuum theory the slip at the wall is exactly zero. From the molecular theory it is known that the slip is extremely small and takes place in a length scale of the order of mean free path. But it cannot be verified by direct observations and also the experimental justifications have been indirect and they partly depend on some kind of modelling, e.g. N-S equations. Additionally the experiments have their own error bands. Hence in the true scientific spirit the question – How much confidence should we place in the no-slip boundary condition? – cannot be brushed aside.

If we accept this question to be pertinent the immediate question that follows is about the validity of the N-S equations themselves. A major assumption in the derivation of the N-S equations is the relation (linear, for Newtonian fluids) between the stress and rate of strain. Our experimental verification of this relation is, at best, in the same class as the experimental ‘proofs’ we have discussed in this article about the no-slip boundary condition. Hence the individual experiments cannot help us to justify the N-S equations. Truesdell [6] gives the similitude arguments to justify the N-S equations and we can extend them in the present context.

It is proper here to consider the N-S equations clubbed with the no-slip boundary conditions to be the model under scrutiny. In many examples the boundary conditions at the wall play a dominant role. Our confidence in this combined system comes from the rules of similitude this system satisfies. Criteria in terms of non-dimensional parameters like the Reynolds number, Mach number etc. have proved themselves valid in a variety of circumstances including in turbulent flows (See *Box 3*). If there were a partial slip according to Navier’s hypothesis another length scale μ/β enters the equations in addition to the length d specifying the system. This length μ/β would have been detected in the similitude tests unless μ/β is zero or so small that its effects are negligible as we have seen. It is these collective

It is the collective experimental observations that give us confidence in the validity of the N-S equations and the no-slip condition at the wall.



Box 3. No-Slip in Turbulent Flows

It was mentioned earlier that Couette concluded that the no-slip condition is valid for the turbulent flows also. Turbulent flows are essentially unsteady and also have a large velocity gradient near the wall. But at every instant they have to satisfy the same wall boundary conditions that a laminar flow does. Turbulent quantities are often decomposed into a time averaged value and a fluctuation; e.g. the instantaneous velocity component $u(x, y, z, t)$ parallel to the wall can be written as a sum of the mean and the fluctuation:

$$u(x, y, z, t) = \bar{u}(x, y, z) + u'(x, y, z, t). \quad (\text{B3.1})$$

If $u(x, y, z, t)$ satisfies the no-slip condition, it is easy to see that so should \bar{u} and u' . Similar arguments apply to the normal velocity component $v(x, y, z, t)$. Hence turbulent velocity fluctuations, no matter how severe, get suppressed at the wall. If the flow near the wall is unidirectional and nearly parallel to the wall, (i.e. boundary layer-type flow) remarkable similarity rules exist for the mean velocity distribution in this flow. Such similarity in the wall layers is very unlikely if there were slip at the wall.

We saw in section on Historical Development that friction at a wall, for example resistance coefficient λ for pipes, does not depend on the wall conditions for laminar flows and also for turbulent flows if the surface is sufficiently smooth. But rough surfaces in turbulent flows lead to a larger friction.

experimental observations that should give us great confidence in the validity of (the N-S equations and) the no-slip condition at the wall.

Suggested Reading

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