The Meaning of Integration – II

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1. Riemann Integration

In Part I, we defined the area under a curve, which is given by a continuous function, as the integral of such a function. In this part, we discuss the definition of the integral of a class of discontinuous functions. This integral has applications in many practical problems.

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[1] Part 1. *Resonance*, Vol.9, No.4, pp.71-78, 2004.

Cauchy's definition of integral can readily be extended to a bounded function with finitely many discontinuities. Let c be a point of discontinuity. We would like to find the area below the graph (see *Figure* 1a) as explained earlier. The idea is as follows. Choose a small number $r > 0$. As the function f is continuous in $[a, c-r]$ and $[c+r, b]$, Area$(A_1) = \int_a^{c-r} f(x)dx$ and Area$(A_2) = \int_{c+r}^b f(x)dx$ are well defined. Area(B_r) becomes smaller as r becomes closer to 0! There is a little ambiguity here as the area of B_r is not defined yet, since there is a discontinuity in $[c-r, c+r]$. This has to be explained in a slightly different way. Using the boundedness of f one sees that the integrals $\int_a^{c-r} f(x)dx$, $\int_{c+r}^b f(x)dx$ converge to unique real numbers k_1, k_2 respectively as r converges to zero. This can be interpreted as saying that Area(B_r) converges to zero. Thus we take $\int_a^b f(x)dx = k_1 + k_2$. The extension of integrals to bounded functions with finitely many discontinuities is almost immediate.

On the other hand, if f is unbounded (*Figure* 1b) then, in general, it is not true that Area(B_r) converges to zero in the above sense. In fact, one or both of the integrals $\int_a^{c-r} f(x)dx$, $\int_{c+r}^b f(x)dx$ can become larger and larger as r becomes smaller and smaller. For example consider the two functions defined on $(-1, 1)$:

$$f_1(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 0 & \text{at } x = 0 \end{cases} \quad \text{and } f_2(x) = \begin{cases} \frac{1}{|x|^{1/2}} & \text{if } x \neq 0 \\ 0 & \text{at } x = 0. \end{cases}$$

Keywords
Riemann integrability, oscillation of functions, Lebesgue measure.

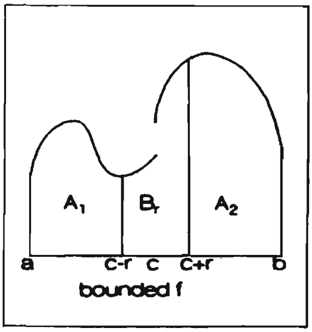


Figure 1a.

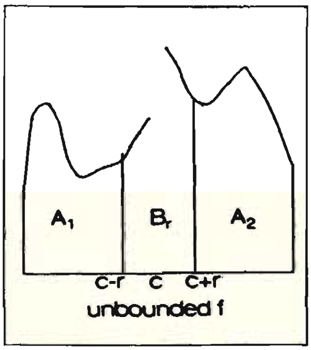


Figure 1b.

Then both are discontinuous at $x = 0$ and unbounded. But the limits as $r \to 0$ of the two integrals $\int_{-1}^{-r} f_1(x)dx$, $\int_r^1 f_1(x)dx$ do not exist, whereas both limits exist in the case of f_2.

We remark that the definition of Cauchy sums given by equation (3) in Part I does not require either the assumption of continuity or any analytical expression of f Cauchy used the continuity to prove that the sum S_P indeed converges to a unique real number. But he did not pursue the research with more general functions. A possible reason is that at that time the concept of a function was in terms of analytical expressions. Perhaps, Peter Gustav Lejeune Dirichlet (1805-1859) was the first mathematician to look seriously at a function as an association $x \to f(x)$ rather than an analytical expression, but he did not succeed in going further; essentially his work was based on continuous functions.

Then arrived the nineteenth century genius Riemann. His starting point was also the Cauchy sums defined for a bounded function, but without any continuity assumption. Given a partition P, form the following sums

$$S_P = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}), \quad \tilde{S}_P = \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1}), \quad (1)$$

where $t_i, s_i \in [x_{i-1}, x_i]$ are such that

$$f(t_i) = \text{minimum } \{f(x) : x \in [x_{i-1}, x_i)\},$$

$$f(s_i) = \text{maximum}\{f(x) : x \in [x_{i-1}, x_i)\} \text{ }^1.$$

Warning: If f is not continuous, the maximum and minimum have to be replaced by sup and inf, respectively. For an arbitrary set B, we define $\bar{b}$ =sup B if $\bar{b} \leq b$ for all b in B and no other number satisfies this property. Similarly for inf.

The sums S_P and $\tilde{S}_P$ represent the areas given by the shaded regions in *Figure* 9a (in Part I) and *Figure* 2 respectively and $S_P \leq \text{Area}(A) \leq \tilde{S}_P$ [2]. Moreover, if P' is a refinement of P, then

$$S_P \leq S_{P'} \leq \text{Area}(A) \leq \tilde{S}_{P'} \leq \tilde{S}_P. \quad (2)$$

[2] This is due to the selection of t_i, s_i.

If f is continuous, then there is nothing special about the points t_i, s_i appearing in (1) and in fact, one can take any point in $[x_{i-1}, x_i]$ and form the corresponding sum. In this case, as was shown in part I, all these sums converge to the same real number, namely $\int_a^b f(x)dx$. The sums $S_P, \tilde{S}_P$ are respectively known as the 'Lower Sum' and 'Upper Sum'.

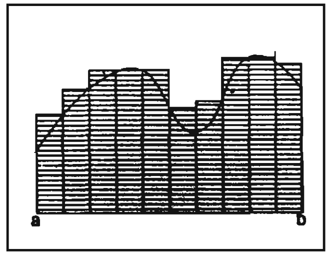


Figure 2.

But if the function is not continuous, the situation is altogether different. However, using the boundedness of f, one can show that $S_P, \tilde{S}_P$ converge as the partition gets finer and finer, that is $|P| :=$ Maximum $\{x_i - x_{i-1}, 1 \leq i \leq n\} \to 0$, to some real numbers, say k_1, k_2 respectively. Then from (2) it follows that

$$k_1 \leq \text{Area}(A) \leq k_2. \qquad (3)$$

It is possible that k_1 and k_2 are different (see *Box* 1) and hence we cannot define the Area(A) or $\int_a^b f(x)dx$ in a unique fashion. Of course, if $k_1 = k_2$, then we have $\int_a^b f(x)dx = k_1 = k_2$ and we are done. In this case we say f is Riemann integrable (R-integrable) over $[a, b]$.

So what have we achieved? Nothing except that we have slightly reformulated the problem. The ingenuity of Riemann is that he gave an equivalent integrability condition and then he presented an example of a function which is discontinuous on a dense set of points. Let us analyze it a bit further. Our aim is to characterize the class of all R-integrable functions. To this end, we have to concentrate on the set of all points of discontinuity of f and mainly the contribution of the jump at a discontinuous point or the oscillation of f in the subintervals.

Let $D_i = \sup_{x,y\in[x_{i-1},x_i]}[f(x) - f(y)] = \sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)$ be the oscillation of f in $[x_{i-1}, x_i]$. Let $\delta_i = |x_i - x_{i-1}|$, $|P| = \max_{1\leq i\leq n} \delta_i$. Then it is easy to see that[3] f is R-integrable if and only if (note that if $|P| \to 0$, then $n \to \infty$)

$$(R_1) \qquad \lim_{|P|\to 0}(D_1\delta_1 + \quad + D_n\delta_n) = 0.$$

[3] Note that $k_1 = \sup_P S_P$ and $k_2 = \inf_P \tilde{S}_P$. Use the definition of sup and inf.

Box 1.

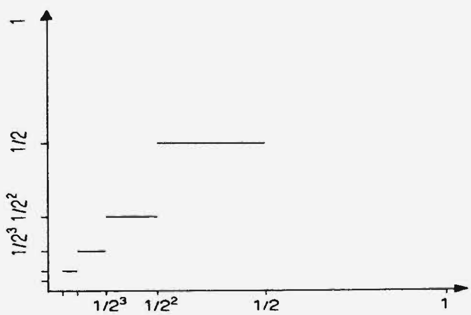
An R-integrable function which is discontinuous at infinitely many points

One can easily construct many examples. For, write $(0,1] = (\frac{1}{2},1]\bigcup(\frac{1}{2^2},\frac{1}{2}]\bigcup\ldots\ldots$ and define

$$f(x) = \begin{cases} 0 & \text{at } x = 0 \\ \frac{1}{2^{n-1}} & \text{if } x \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}], n = 1, 2, \ldots.. \end{cases}$$

For any fixed n, the function f is R-integrable in the interval $[\frac{1}{2^n}, 1]$ as f has only finitely many discontinuities in that interval. Moreover $\int_{\frac{1}{2^n}}^{1} f(x)dx = \sum_{k=1}^{n} \frac{1}{2^{k-1}}(\frac{1}{2^{k-1}} - \frac{1}{2^k}) = \sum_{k=1.}^{n} \frac{1}{2^{2k-1}}$ which converges to $\frac{2}{3}$ as $n \to \infty$.

If P is any partition (assume the point $\frac{1}{2^n}$ is a point on the partition), then the contribution to the sums $S_P = \tilde{S}_P$ from the interval $[0, \frac{1}{2^n}]$ can be made arbitrarily small by choosing n large enough. Thus f is R-integrable in $[0,1]$ and $\int_0^1 f(x)dx = \frac{2}{3}$.



Fig(13)

Riemann then replaced the above condition by an equivalent condition which characterizes the class of all R-integrable functions in terms of the set of discontinuities.

Let $d > 0$ be any positive number. Look at all those partitions P such that $|P| \leq d$ and let

$$\Delta = \Delta(d) = \max_{|P| \leq d}(D_1\delta_1 + \ldots + D_n\delta_n).$$

Then f is integrable if and only if $\Delta(d) \to 0$ as $d \to 0$. Now for a given partition P, the idea is to get hold of those intervals for which the oscillations are high. (Note that at a point of continuity the oscillation goes to zero.)

For this purpose, let $\sigma > 0$ and $L = L(P, \sigma)$ denote the sum of the δ_i's for which D_i is greater than σ. The integrability of f is then equivalent[4] to the following condition (R_2).

$$(R_2)\begin{cases}\text{Corresponding to every pair of positive numbers } \varepsilon \text{ and } \sigma, \text{ there exists a positive number } d \\ \text{such that if } P \text{ is any partition with } |P| \leq d, \\ \text{then } L(P, \sigma) < \varepsilon\end{cases}$$

If you are familiar with the theory of Lebesgue measure, we can say that f is R-integrable if and only if the set of points of discontinuity has Lebesgue measure zero. In simple terms, it means that the set of all points of discontinuity is contained in a countable collection of open intervals whose total length can be made arbitrarily small. Two examples of R-integrable functions with infinitely many discontinuities are given in *Box* 1 and *Box* 2.

2. A Quick Look at the Lebesgue Measure and Integration

As remarked earlier, Riemann's condition for integrability is so weak that one would consider it to be the most general form of integration. But R-integration was not powerful enough to handle many important problems from analysis, for example, the interchange of limit and integration which was an often needed tool in analysis. A second problem was the lack of 'completeness'(see *Box* 3) in addition to other questions of Fundamental Theorem of Calculus, etc. It is in this respect that the measure theoretic ideas became very important as they provided a new basis for defining integrals not only in the 'Euclidean Space' (line, plane, space, etc.), but in more general spaces as well.

The originality of Riemann was in his approach of representing functions by trigonometric series. He was able

[4] The contribution from the intervals for which $D_i > \sigma$, to the sum $\sum_{i=1}^{n} D_i\delta_i$ is at least σL. Therefore $\sigma L \leq \sum_{i=1}^{n} D_i\delta_i \leq \Delta = \Delta(d)$ and $L \leq \frac{\Delta}{\sigma}$. Consequently, if f is integrable, then $L \to 0$ as $d \to 0$, which is nothing but (R_2). Conversely, if (R_2) is true, then for given $\varepsilon > 0, \sigma > 0$, $\exists\, d$ as in (R_2). Now for any partition P with $|P| < d$, we have $\sum_{i=1}^{n} D_i\delta_i \leq DL + \sigma(b-a) \leq D\varepsilon + \sigma(b-a)$, where D is the oscillation of f on $[a, b]$, which is finite, since f is bounded. (R_1) then follows, since ε and σ are arbitrarily small.

Box 2.

An R-integrable function with discontinuities on a dense set

For any real number x, let $m(x)$ be the nearest integer to x and let

$$g(x) = \begin{cases} 0, \text{if } x = \frac{n}{2}, n \text{ odd} \\ x - m(x) \text{ otherwise.} \end{cases}$$

Then $g(x)$ is discontinuous at the point $x = \frac{n}{2}$, n odd. Riemann, then defined the function

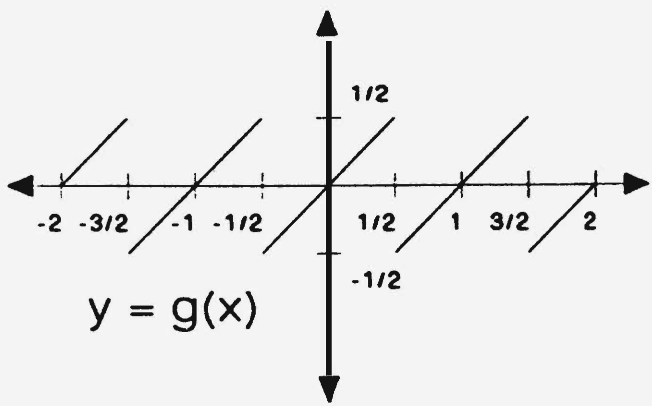
$$f(x) = g(x) + \frac{g(2x)}{2^2} + \ldots + \frac{g(nx)}{n^2} + \ldots \ .$$

The function f is discontinuous at any point of the form $\frac{m}{2n}$ where m and n are relatively prime. In fact, for these values of x,,the right hand and left hand limit of f at x are

$$f(x+) = f(x) - \frac{1}{2n^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(x) - \frac{\pi^2}{16n^2},$$

$$f(x-) = f(x) + \frac{1}{2n^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = f(x) + \frac{\pi^2}{16n^2}.$$

Since the set $E = \{\frac{m}{2n} : m, n \text{ relatively prime }\}$ is dense*, we see that f is discontinuous on a dense subset of real numbers. But f is R-integrable on any finite interval because for any $\sigma > 0$, there are only a finite number of points $x = \frac{m}{2n}$ in any finite interval at which the jump, $f(x-) - f(x+) = \frac{\pi^2}{8n^2} > \sigma$, and the condition (R_2) is consequently satisfied.



* A set X is dense in the set of real numbers if any reals number can be approximated by elements from X. Let x be any real number and since rationals are dense in reals, the number $2x$ can be approximated by numbers of the form $\frac{m}{n}$. Thus x can be approximated by numbers of the form $\frac{m}{2n}$.

Box 3.

An sequence of R-integrable functions whose limit is not R-integrable

Let A and B, respectively be the set of all rationals and irrationals in $[0,1]$. Since A is countable, we can arrange them as $A = \{r_1, r_2, ...r_n, ...\}$. Let f_n be the characteristic function of the set $\{r_1, ...r_n\}$, i.e.,

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_i, \text{ for some } i = 1, ...n \\ 0 & \text{otherwise.} \end{cases}$$

Since f_n's are discontinuous exactly at $r_1, ...r_n$, it follows that f_n's are R-integrable. It is easy to see that the pointwise limit f of f_n is the characteristic function of A, i.e.,

$$f(x) = \begin{cases} 1 & \text{if } x \epsilon A \\ 0 & \text{if } x \epsilon B. \end{cases}$$

The function f is not R-integrable. To see this, let $P := 0 = x_0 < x_1 < .... < x_n = 1$ be any partition of $[0,1]$. Since any subinterval $[x_{i-1}, x_i]$ contains both rationals and irrationals, we get $S_P = 0$ and $\tilde{S}_P = 1$. Thus $k_1 = lim_{|P|\to 0} S_P = 0$ and $k_2 = lim_{|P|\to 0} \tilde{S}_P = 1$.

to obtain necessary and sufficient conditions for the representability of a function at a point by a trigonometric series.

In this section, we briefly describe some reasons for the development of Lebesgue integration (L-integration). A generalization of R-integration became necessary when it failed to satisfy mathematicians and other scientists in their problems. One of them is the term by term integration in an infinite series. Under what conditions does the equality $\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$ hold? This does hold if the f_n's are R-integrable and the infinite series converges uniformly. 'Uniform convergence' is so strong that one cannot expect it to hold in practical situations. It all began in the early nineteenth century (1800 - 1805) when Joseph Fourier (1768 - 1866) introduced the 'Fourier' series while studying the heat problem. If $u(x,t)$ represents the temperature at a point x at time t of a thin bar of length l of some conducting material, then Fourier deduced that it satisfies the equation

$$u_t = k u_{xx}, \qquad (4)$$

where, k is a positive constant and $u_t = \frac{\partial u}{\partial t}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ are the partial derivatives of u. If the temperature at both end points of the bar are maintained at zero and the initial distribution is also given, then Fourier obtained the solution of (4) as an infinite series. In the process, he used term by term integration to obtain the coefficients in the infinite series[5].

[5] The Fourier theory was before Riemann and hence the integration was understood via the antiderivative.

Of course, he did not justify the term by term integration and it took more than a hundred years to justify it for some reasonable functions. Fourier was satisfied with his theory as he could verify his results with observations and in fact, he provided many examples. To a certain extent, Fourier was right as the concept of function itself was not clear at that time and proper convergence of functions was not available.

Riemann also attempted to justify the result of term by term integration (equivalently, interchange of limit and integrals) using the broader concept of integrability. Once the uniform convergence was available, the term by term integration was justified. The result in general, is not true if the convergence is pointwise. One major drawback in the Riemann theory is that the pointwise limit of R-integrable functions need not be an R-integrable function (see *Box* 3).

There were many other problems that were left unresolved after the development of Riemann's theory and Lebesgue's theory came as a boon to resolve those. Observe that while forming the Cauchy Sum, we used the area of rectangles. The development of Lebesgue theory was based on the broader concept of measurable sets and measure. In the process he introduced the concept of measurability and Lebesgue Measure. We hope to present these details in a later article.

The R-integrability condition was obtained by looking at the oscillation at discontinuous points. Therefore, if the oscillation is large for 'too many' [6] points, then we may not be able to make $\Delta(d)$ small as d goes to zero and so the function will not be R-integrable. For example, for the Dirichlet function (characteristic function of $\mathbf{Q}\bigcap[0,1]$), $D_i = 1$ for all i and thus $\Delta(d) = 1$.

[6] the term 'too many' is vague, but can be obtained from the R-integrability condition.

Now observe that oscillation is a measurement in the range of the function. Thus, if we wish to render non R-integrable functions as integrable functions in some sense, follow the simple philosophy: "strike where it hurts the most" In other words, split the range into smaller intervals. To be more precise, let $f : [0,1] \mapsto \mathbf{R}$ be bounded and so let $-M \leq f(x) \leq M$ Divide the interval $[-M, M]$ into $2n$ equal parts $I_i = [\frac{iM}{n}, \frac{(i+1)M}{n}]$, $-n \leq i \leq n-1$. For large n , these intervals are small. Now look at the pre-image sets $E_i = f^{-1}(I_i)$ which are disjoint subsets of $[0,1]$. Further, the sets E_i's form a partition of $[0,1]$ and the oscillation of f on each E_i is small and in fact, it is $\leq \frac{1}{n}$.

There are two issues. First of all, the sets E_i need not be intervals or unions of intervals and secondly, even if they are so, the intervals need not be small, even if we choose n sufficiently large. The second case is not a major issue as these intervals can be split into smaller ones which will not increase the oscillations. The first case is a non trivial issue and led to the concept of associating a 'length' or in modern language a 'measure' to an arbitrary set E. Suppose we could define $length(E_i) = l(E_i)$ in some way, then we can define lower and upper sums in a similar way as earlier, but for a broader class of partitions than partitions by intervals. Thus, if we define, $\bar{k}_1$ = sup of lower sums and $\bar{k}_2$ = inf of upper sums, where the sup and inf are taken from the broader class of partitions, we get

$$k_1 \leq \bar{k}_1 \leq \bar{k}_2 \leq k_2.$$

Thus even if $k_1 \neq k_2$, it can happen that $\bar{k}_1 = \bar{k}_2$ which gives rise to a broader concept of integrability. Hence a generalization of R-integrability is based on a broader concept of defining appropriate 'lengths' to subsets of $\mathbf{R}$.

Therefore the primary question reduces to: Is it possible to associate a length or measure to arbitrary subsets preserving certain natural properties? It is not to be so, but Lebesgue has succeeded in getting a large class of sets known as 'Lebesgue measurable sets' by imposing certain restrictions. This is achieved by the well known 'measurability' condition. 'Measure theory' deals with such questions (see [4,5,6]).

Suggested Reading

[1] E T Bell, *Men of Mathematics*, Simon & Schuster, New York, 1965.

[2] Richard Courant and Herbert Robbins (Revised by Ian Stewart), *What is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford University Press, 1996.

[3] Thomas Hawkins, *Lebesgue's Theory of Integration*, Chelsea Publishing Co., The Bronx, New York, 1975.

[4] H L Royden, *Real Analysis*, Prentice Hall of India, New Delhi, 1995.

[5] Walter Rudin, *Real and Complex Analysis*, Tata McGraw-Hill Publishing Co., New Delhi, 1983.

[6] Walter Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1989.

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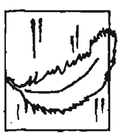

If we indulge in fanciful imagination and build worlds of our own, we must not wonder at our going wide from the path of truth and nature ... On the other hand, if we add observation to observation, without attempting to draw not only certain conclusions, but also conjectural views from them, we offend against the very end for which only observations ought to be made.

– William Herschel