The No-Slip Boundary Condition in Fluid Mechanics

1. The Riddle of Fluid Sticking to the Wall in Flow

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A moving fluid in contact with a solid body cannot have velocity relative to the body. Even though the question whether there is slip has been satisfactorily resolved now, it was a difficult and controversial problem. In the first part of this article several basic ideas and details related to this problem are discussed. The concluding part of the article will trace the development of ideas leading to the resolution of this important question. Extreme cases where fluid does slip will also be discussed.

1. The No-Slip Boundary Condition

It is known now, beyond any doubt, that a moving fluid in contact with a solid body will not have any velocity relative to the body at the contact surface. This condition of not slipping over a solid surface has to be satisfied by a moving fluid. This is known as the no-slip condition and is stated routinely in textbooks on fluid mechanics (see Goldstein [2]). But it remained a difficult problem for a long time. We will first give in this article some basic ideas connected to this problem so that the historical notes added afterwards will be appreciated better. In the next part some recent experimental data of interest, the phenomenon at the molecular level and the case of turbulent flows will be discussed briefly. We will see that this simple looking phenomenon was so difficult to comprehend and even the giants had to struggle. The students today are taught it in one stroke. It is not surprising that some of them get bothered about it. If they did not, we will see why they should be.

Keywords
No-slip boundary condition, Navier-Stokes equations, Poiseuille flow, flow resistance.

2. What Happens to a Fluid Particle at the Wall?

Before considering the case of fluids, i.e. gases and liquids,

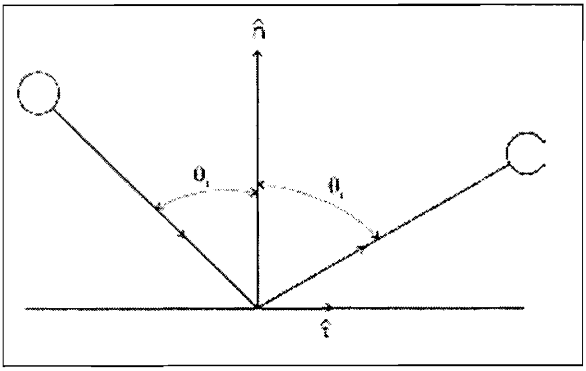
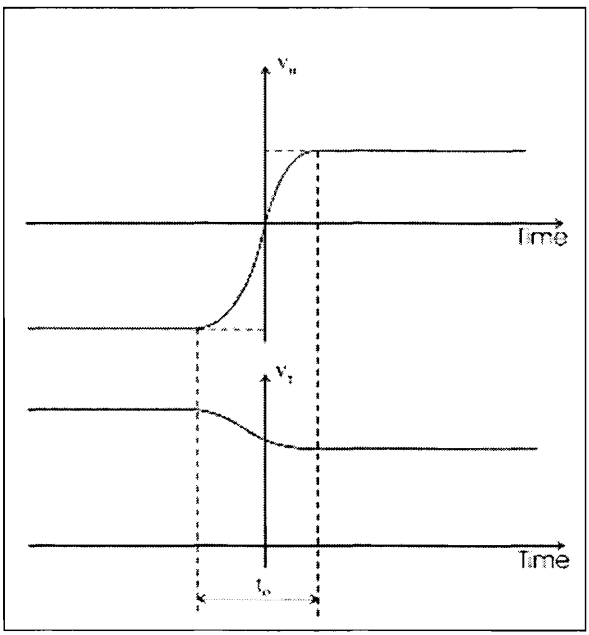


Figure 1. A ball impinging on a wall.

consider a simpler case of an isolated solid ball. When the ball hits a wall of a solid body, its velocity abruptly changes. This abrupt change in momentum of the ball is achieved by an equal and opposite change in that of the wall or the body. Thus the overall momentum is conserved in each of the three directions.

Now assume that the ball is spherical and nearly rigid and it impinges on a smooth rigid wall at an angle (see *Figure* 1). If the wall is heavy its motion can be neglected. At the point of collision we can identify normal and tangential directions $\hat{n}$ and $\hat{\tau}$ to the wall. The time of impact t_0 is very brief. It is a good assumption to conclude that the normal velocity V_n will be reversed with a reduction in magnitude because of loss of mechanical energy. If we assume the time of impact to be zero, the normal velocity component V_n is seen to be discontinuous and also with a change in sign (see *Figure* 2). Whether it is discontinuous or not, the fact that it has to change sign is obvious, since the ball cannot continue penetrating the solid wall. The case of the tangential component V_τ is far more complex and more interesting. First of all, the ball will continue to move in the same direction and hence there is no change in sign. If the wall and the ball are perfectly smooth (i.e. frictionless) V_τ will not

Figure 2. Time variation of normal and tangential velocity components of the impinging ball.



Box 1. A Solid Body Rolling With or Without Slip

A ball can also roll on the surface. If we assume the ball to be spherical and rolling on the surface without slip the contact point is the instantaneous centre of rotation and the contact point is not moving at that instant. The contact point keeps changing as the ball rolls. Friction is required to avoid slip but it does not assure that slip is completely prevented. There is no constraint from any theoretical considerations for the slip to be present or to be zero. One can imagine that if the velocity increases there is likely to be some slip. This has some implication in understanding fluid motion near the wall.

change at all. But in case of rough surfaces V_τ will decrease a little. But it is important to note that V_τ is nowhere zero. This is true even when we relax the assumptions made in this model. Even though the ball sticks to the wall for a brief period t_0, at no time is its tangential velocity zero!

The ball can also roll on the wall. See *Box* 1.

The problem of fluids is considered now. This is fundamentally different from the case of an isolated ball since a *flow field* has to be considered now. The difference is that a fluid element in contact with a wall also interacts with the neighbouring fluid. Once we recognize this difference, the problem of velocity boundary condition at the wall appears too difficult and the previous model of an isolated ball impinging or rolling on the wall is not of much help. It is not surprising that only at the end of the 19th century was this problem resolved using both theoretical and experimental tools. During the whole of that century extensive work was required to resolve the issue.

The problem of fluid coming in contact with the wall is fundamentally different from the case of an isolated impinging ball. A *flow field* has to be considered now.

Even though we agreed that the case of a simple ball is not adequate here, an idea used there can still be applied here. The idea is that the normal component of velocity at the solid wall should be zero to satisfy the no penetration condition. Quite interestingly in the case of fluids the tangential velocity is also zero at the wall. This is the so-called no-slip boundary condition and we will see how different it is compared to the simple ball impingement case. Before giving the details and a historical perspective some details of the fluid motion are given in the next section.

3. Continuum Hypothesis and the Navier–Stokes Equations

Since the number of molecules in a given fluid volume is very large, it is possible to ignore the existence of the individual molecules and consider the fluid to be homogeneous and of uniform properties. This is much simpler than considering the dynamics of the molecules and this is the so-called continuum hypothesis. In this model the fluid does not have any voids like intermolecular spaces. The classical laws of motion, of course, apply to this fluid. We talk of the fluid elements or fluid particles which deform in the flow. Forces acting on these elements determine the acceleration. But the forces consist of both the externally applied forces like those due to gravity or magnetic field and also the internal stresses including pressure. The stresses acting on a fluid element are determined by the *rate* of deformation of the element. This is where one faces the difficulty. How does one relate stresses to the rate of deformation or velocity components?

This relation between stress and rate of deformation was obtained independently in the first half of the 19th century by the French engineer Navier (1785-1836) and the Irish mathematician and physicist Stokes (1819-1903). They derived the well-known equations of motion known now as the Navier–Stokes equations which relate the acceleration of the fluid element to the net force acting in each direction. We need the appropriate boundary conditions to solve these equations. Quite interestingly these equations helped in resolving the uncertainty about the no-slip boundary condition. These equations will be given in the next section.

Two other ideas are relevant here. The first is about the nature of fluid stress and is discussed in *Box* 2. The second idea is about the specification of the velocity field and is discussed in *Box* 3.

Box 2. Stress in a Fluid

When the fluid is at rest, only the normal stresses are exerted, the tangential stresses being zero. The normal stress at a point does not depend on the direction and it is the hydrostatic pressure. When the fluid is in motion, the pressure changes from this hydrostatic value and also additional tangential stresses are induced. It suffices for the present purpose to know that these additional stresses are obtained by multiplying the *rate* of deformation (which is related to spatial velocity derivatives) by the viscosity of the fluid. For common fluids like water and air this *linear* relation between the stress and rate of strain or deformation is a good approximation and such fluids are known to be newtonian.

4. Some Details and Simplifications

Even though we described the N–S equations in the previous

Box 3. Lagrangean and Eulerian Description of a Velocity Field

Two alternatives are possible. In the first, or the so-called Lagrangean description, we extend the idea from particle mechanics. Here the velocity is associated with distinct pieces of matter that are identified like a particle or ball. In the second description, known as the Eulerian description, the velocity is associated with a location in the flow but not any distinct matter. Hence when we say x-component of velocity u at location (x,y,z) and time t i.e. $u\,(x, y, z, t)$ it pertains to a location and hence to different pieces of fluid occupying this location at the instant considered. Both these descriptions of velocity are used in the study of fluid motion depending on the context, but the Eulerian description is more common. This directly gives the spatial gradients needed to calculate the stress. A fixed probe meant to measure the velocity like a Pitot tube or a hot-wire probe or a laser Doppler anemometer measures the Eulerian velocity. We will use this description only.

section, their mathematical form was not given. It is possible to read this article without considering this exact form given below. However, considering these details will be more fruitful. Since the N–S equations are solved along with the mass conservation or the continuity equation we give that equation also (see (1) below). To simplify matters we consider flow only in two directions (x, y) Let t be time, (u, v) the velocity components along (x, y) and p be the pressure. Further the fluid is assumed to be incompressible with density ρ and viscosity μ. Then

$$u_x + v_y = 0 \quad (1)$$

$$\rho\,(u_t + uu_x + vu_y) = -p_x + \mu(u_{xx} + u_{yy}) \quad (2)$$

$$\rho\,(v_t + uv_x + vv_y) = -p_y + \mu(v_{xx} + v_{yy}). \quad (3)$$

Here the subscripts indicate partial differentiation. Hence $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $v_{xx} = \partial^2 v/\partial x^2$ etc. In equations (2) and (3) the left hand side represents the acceleration of a fluid element and the right hand side the net force on it.

If we neglect viscous terms in the N–S equations we get the Euler equations. These equations are still nonlinear.

This system of three equations has three dependent variables (u,v,p) and can be solved if appropriate boundary conditions are specified. Specification of the boundary conditions is a mathematically difficult issue and we will not deal with that in detail. Also, notice that this system is second order in space because of terms like u_{xx} on the right hand side and is non-linear because of the nonlinear (or second power in dependent variables) terms like uu_{xx} on the left hand side.

If we neglect the viscosity of the fluid the second order terms in equations (2) and (3) will be dropped. Then what are left are the Euler equations of motion. These inviscid equations are first order in space and are still non-linear. These were derived by the Swiss mathematician Euler (1707-1783) before Navier and Stokes gave the equations for a real (that is viscous) fluid.

An interesting observation about this simplification: When we drop the viscous terms to get the Euler equations, the order of the equations decreases by one. This should also translate into a reduced requirement on the boundary conditions. And that is exactly what happens. For the Euler equations we specify only the normal component of the velocity (for example, to be zero at a stationary wall). The solution of the Euler equation can lead to a slip velocity at the wall. For the N–S equations, which are of one order higher, we have to specify the tangential component also. Note that there is no need for it to be zero to solve the equations but its precise description is required. For an interesting discussion of this point see Arakeri and Shankar [1].

5. The Hagen-Poiseuille Flow

This is the fully developed laminar flow in a long tube of circular cross-section. We discuss it here since it will be referred to frequently in the rest of the article and also because it was very helpful in the experimental verification of no-slip. The need for specification of the tangential velocity on the wall will be specially highlighted. For mathematical simplicity we consider a 2-D planar flow rather than flow through an axisymmetric tube. The mathematical details and the qualitative results are similar in both the cases. Final results will be given for the case of the axisymmetric tube also. Mathematically-minded readers will be benefited by some of the details given below. Those who are not interested in the details can go straight to the final results in this section.

To solve the N–S equations we have to apply the tangential boundary condition also. Quite interestingly they helped to resolve an uncertainity about the boundary condition.

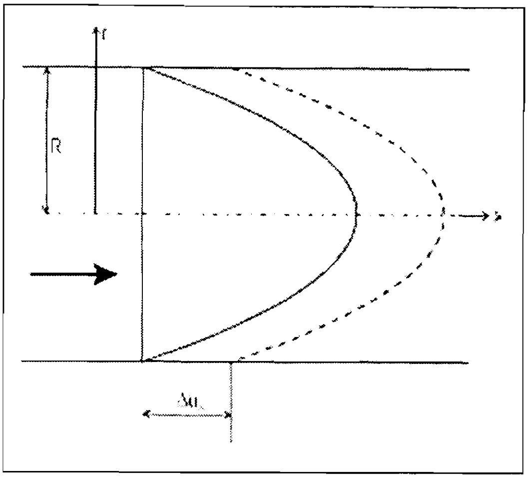


Figure 3. Parabolic velocity profile in a fully developed pipe flow with and without slip.

In a steady flow all derivatives with respect to time t and in a very long tube far away enough from the entrance all derivatives with respect to x will go to zero (see *Figure* 3 which is shown for a tube of radius R rather than a channel of half-width h; in the channel case r is to be replaced by y). Hence from equation (1) $v_y = 0$ leading to $v=$ constant. This constant is zero since v, the normal velocity on the wall at $y = h$ is zero. Now each of the terms on the LHS of equations (2) and (3) is zero. And equation (3) reduces to $p_y=0$ leading to $p=p(x)$, i.e. in the entire cross-section p is constant. Equation (2) simplifies to $p_x=\mu u_{yy}$. Notice that p_x is the total derivative dp/dx and further since u_{yy} cannot depend on x, p_x should be independent of x, and hence a constant. This equation can be integrated twice. It is here that we have to specify the tangential velocity on the wall. Whether there is slip or no-slip is unimportant in the solution of the equation but its precise specification is mandatory. This flow gives us an excellent opportunity to measure the slip if there is any.

Integrating $p_x=\mu\, u_{yy}$ w.r.t. y we get

$$u_y = \frac{p_x}{\mu} y + A, \tag{4}$$

$$u = \frac{p_x}{2\mu} y^2 + Ay + B \tag{5}$$

Because of symmetry in y, $A=0$. The other constant B is fixed by the value of u on the wall, at $y=h$. If Δu_w is the assumed slip at the wall,

Discharge in a tube increases for the same pressure drop if fluid slip were present at the wall.

$$B = \Delta u_w - \frac{p_x}{2\mu} h^2 \tag{6}$$

$$u = -\frac{1}{2\mu}\frac{dp}{dx}(h^2 - y^2) + \Delta u_w. \tag{7}$$

Since pressure is decreasing along x, p_x is negative.

The corresponding equations for a tube (see *Figure* 3) of radius R are:

$$u = -\frac{1}{4\mu}\frac{dp}{dx}\left[R^2 - r^2\right] + \Delta u_w \tag{8}$$

$$u_r = \frac{r}{2\mu}\frac{dp}{dx} \tag{9}$$

$$Q = 2\pi\int_0^R ur\,dr = \pi\left[\frac{R^4}{8\mu}(-p_x) + R^2\Delta u_w\right], \tag{10}$$

where Q is the flow rate. Note that the presence of velocity slip at the wall does not change the shear stress distribution but the discharge Q increases due to slip for a given pressure drop.

It is very tempting to conclude by looking at these equations that we can measure the slip or at least decide whether the slip is there at all. But it is not so simple. If we define the resistance coefficient λ for the tube by

$$\lambda = (-p_x)\frac{2R}{\frac{1}{2}\rho\bar{u}^2} \tag{11}$$

and Reynolds number $Re = 2R\rho\bar{u}/\mu$, where the average velocity $\bar{u} = Q/(\pi R^2)$, we get

$$\lambda = \frac{64}{Re}\left(1 - \frac{\Delta u_w}{\bar{u}}\right). \tag{12}$$

A graph of λ vs. Re on a log-log plot with $\Delta u_w = 0$ is a straight line with slope $=-1$ (*Figure* 4). A constant value of $\Delta u_w/\bar{u}$ will shift the graph but the slope will remain constant. Hence it will be difficult to identify if the slip is present.

It may be added that Poiseuille was led to this study to understand the blood flow and recommended that the hydraulic engineers should study the motion of particles in moving liquids with the aid of a

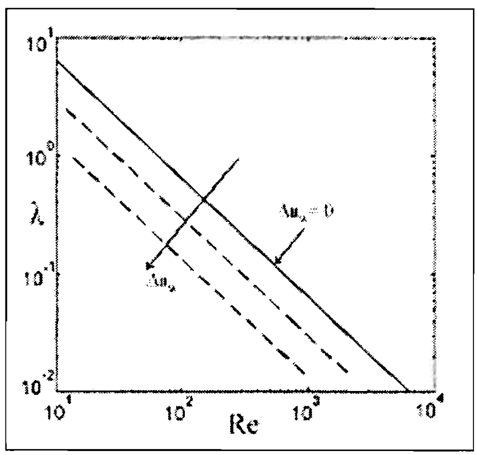


Figure 4. Variation of resistance coefficient λ as a function of Re for a fully developed pipe flow with and without slip.

microscope (1846). Hagen assumed zero velocity at the wall in an earlier paper but later (1839) adopted the idea of a stagnant layer near the wall but without slip.

6. Dry Friction

It is appropriate to recapitulate here our knowledge about the dry friction that occurs when a solid surface slides over another dry solid surface. Conceptually the ideas in this case may appear simpler compared to the fluid case but getting reliable quantitative data is very difficult. The surface conditions affecting dry friction may not be uniform and may even depend on the direction of motion. On the other hand, fluid properties for simple fluids like air and water are more uniform and experiments with the fluid friction become more repeatable.

In the light of the comments made above, it is not surprising that the laws of dry friction were not developed too early in the human history even though it is likely that many had an intuitive feeling for them. It is the experiments of Coulomb in 1781 and those of Morin from 1831-1834 that played a decisive role in formulating the laws of dry friction or Coulomb friction. This period roughly coincides and slightly precedes the time when the laws of fluid friction were also being developed.

The laws of dry friction were not developed too early in the human history even though it is likely that many had an intuitive feeling for them.

Imagine a solid block kept on a table and pushed gently sideward. When a dry surface has a tendency to slide over another similar surface held fixed, the normal forces on these two bodies at the contact surface balance each other. Also, the tangential force or the friction opposes the motion or the tendency to move. One can imagine that at the micro level the two surfaces are bound to have irregular ridges and valleys and thus contact each other only at select locations. The tendency of motion is opposed by these micro-irregularities and the opposing force at this level is not necessarily along the mean contact surface. The net frictional force is equal to the applied force as long as it is less than a limiting force and hence the body does not move. This limiting value is the maximum of static friction at impending motion. If the applied tangential force is greater than this threshold value, the body will start moving but then the frictional force known as the kinetic friction, is slightly less than the maximum static friction. Note that the contact locations at the micro level are continuously changing. One can imagine that when in motion it is the top parts of these micro-ridges that are in contact and this leads to a smaller tangential force. This results in the kinetic friction being slightly smaller than the maximum static friction.

The maximum static friction is independent of the area of contact and is proportional to the normal contact force N between the two surfaces:

$$F_s(\max) = \mu_s N \tag{13}$$

where μ_s is the coefficient of static friction. Similarly for the case of sliding we define the coefficient of kinetic friction μ_k by

$$F_k = \mu_k N \tag{14}$$

Generally μ_k is less than μ_s as mentioned above.

Polishing a surface generally leads to a decrease in the dry friction. We will see some interesting contrast between this and the fluid friction.

7. Newton's Slip

One cannot imagine that a curious person like Isaac Newton would not have bothered about the motion of the fluids. He considered some discrete cases of fluid motion. In the three books in the *Principia* (1725) Newton dealt with vortex motion briefly in Book 2. His motivation was to see if the motion of a fluid vortex was consistent with the Keplerian planetary motion. Hence he considered only the circular motion [3]. He was handicapped by not having the governing equations to describe the motion of either idealized or real fluids. But he did recognize that fluid resistance arose due to the velocity *difference* between two spatially separated points. The velocity difference is equivalent to velocity derivative in simple cases. Now we know that it

Newton's motivation to study the fluid vortex motion was to see if it is consistent with the Keplerian planetary motion.

is the rate of strain or rate of deformation that causes stress and the fluid resistance.

To study the vortex motion he considered an infinitely long circular cylinder immersed in an unbounded fluid and rotating about its axis at a uniform speed. The fluid is set into motion by the moving cylinder and the resulting streamlines are circular. Newton dealt with this problem from first principles (i.e. not starting with any ready made equations) with the tacit assumption that there was no fluid slip at the cylinder wall. Unfortunately he obtained an incorrect expression for the velocity distribution. Still his conclusion that the motion of this vortex due to a rotating cylinder (also due to a sphere that he studied in the subsequent proposition) is not consistent with the Keplerian planetary motion turned out to be correct i.e. the velocity distribution along the radius in the vortex and that of the planets in the solar system were not the same.

As mentioned above Newton correctly assumed that a rotating cylinder imparts the velocity to the fluid that is in contact without any slip. However, he missed a similar assumption in the case of a projectile modelled by a cylinder moving forward in the direction of its length. He concluded that the resistance to motion depends on the diameter (this part is correct) but not on the length of the cylinder. This erroneous conclusion that resistance is independent of length has the assumption that there is complete slip, i.e. the curved surface of the cylinder moves without affecting the fluid motion whatsoever. We should keep in mind that even if Newton had assumed that there was no-slip or only partial slip at the cylindrical surface it would not have been easy for him to get a relation for the drag dependence on the length of the cylinders. But it is very likely that he would have then guessed correctly that the drag would increase with the length.

This historical note is added to emphasize how difficult it was to understand the motion of a fluid in contact with a solid body. In the next part we will discuss how the question of boundary condition at a fluid-solid interface was finally resolved.

Suggested Reading

[1] J H Arakeri and P N Shankar, Ludwig Prandtl and Boundary Layers in Fluid Flow, *Resonance*, Vol.5, No.12, pp. 48 - 63, 2000.

[2] S Goldstein (ed), *Modern developments in Fluid Dynamics*, Vol. II, Oxford: Clarendon Press, 1957.

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