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When is a Decimal Expansion Irrational ?

Everyone learns in school that $\sqrt{2}$ is irrational. This, along with Euclid's proof of infinitude of primes, is probably the first time one encounters a proof by contradiction. Most students know that the value of $\sqrt{2}$ is approximately 1.414 but, more often than not, this aspect is not pursued further. What is meant by a number expressed in its decimal expansion being irrational, is not discussed. For instance, the decimal $0.999\cdots$, where 9 recurs indefinitely is understood (after some persuasion perhaps) to be none else than the number 1. The problem here is that the concept of limit takes some time to sink in. Given that start, they can see easily that numbers like $0.\overline{142857}$ where the digits overlined recur indefinitely, are rational. Even if the recurring string occurs after an initial string (for example, a decimal expansion like $27.\overline{142857}$), it still gives us only a rational value because it is the sum of a geometric series with a ratio of the form 10^k.

It is not hard to prove that this is a necessary condition as well, that is, *a decimal expansion of a real number represents a rational number if, and only if, after the decimal place, there is a finite (possibly empty) string after which the digits consist of a finite string (possibly consisting entirely of zeroes) recurring indefinitely.*

Thus, for instance, the number $.101001000\cdots$, where the number of zeroes keeps increasing by 1 has to represent an irrational number. However, from the decimal expansion, sometimes it is not clear whether there is such an eventual recurrence or not. This could be due to our present state of knowledge. For instance, one could define a number $0.1010\cdots$, where the number of zeroes occurring at the n-th step is either increased by one or kept the same depending on whether the number

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The decimal $0.1234\cdots$, where the natural numbers are written in sequence, is clearly irrational since, for instance, the number of zeroes occurring in the powers of 10 keeps increasing. The decimal $0.235711\cdots$, where the set of primes is written down in sequence, is also irrational. This is because there is a prime of the form $10^n a+1$ for an arbitrary n – a special case of Dirichlet's theorem on prime numbers in arithmetic progression. Here is an elementary proof of a general result of this kind; it is essentially proved in [1].

Consider a decimal $x = 0.a_1a_2\cdots$, where $\{a_n\}$ is a strictly increasing sequence of natural numbers having the property that $\sum_n \frac{n^r}{a_n^s}$ diverges for some $r, s > 0$. Then x is irrational.

Note that since the reciprocals of primes do not sum to a finite quantity, this result also implies that $0.23571113\cdots$ is irrational.

Proof

Let, if possible, x be rational. Then, by throwing out some of the first a's and scaling, we may assume that the decimal is actually periodic and not just eventually periodic. Let t denote a period. Let $N_1 < N_2 < \cdots$ denote the natural numbers representing the different numbers of digits occurring for the a_i's. Let d_i denote the number of a_i's which have exactly N_i digits. In other words, $a_1, \cdots, a_{d_1}$ are the a's with N_1 digits and, for each $i \geq 1$, the numbers

$$a_{d_1+\cdots+d_i+1}, \cdots, a_{d_1+\cdots+d_i+d_{i+1}}$$

are the a's which have exactly N_{i+1} digits. Let us write for simplicity $d_0 = 0$. Now, if some d_{i+1} were bigger

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than t, then the numbers

$$a_{d_1+\cdots+d_i+1}, \qquad , a_{d_1+\cdots+d_i+t+1}$$

would all have N_{i+1} digits. Since the length of the string

$$(a_{d_1+\cdots+d_i+1}) \qquad (a_{d_1+\cdots+d_i+t})$$

is tN_{i+1} which is a multiple of t, it follows that

$$a_{d_1+\cdots+d_i+1} = a_{d_1+\cdots+d_i+t+1}.$$

This is a manifest contradiction of the assumption that $\{a_n\}$ is an increasing sequence. Hence, we have shown that each d_i is $\leq t$.

Now, we also have the evident inequalities

$$a_{d_1+\cdots+d_i+d_{i+1}} \geq \qquad \geq a_{d_1+\cdots+d_i+1} \geq 10^{N_{i+1}-1}$$

since these numbers have N_{i+1} digits. We shall show that $\sum_{n=1}^{\infty} \frac{n^r}{a_n^s}$ converges. Now, for each $i \geq 0$,

$$\sum_{j=d_1+\cdots+d_i+1}^{d_1+\cdots+d_i+d_{i+1}} \frac{j^r}{a_j^s} \leq \sum_{j=d_1+\cdots+d_i+1}^{d_1+\cdots+d_i+d_{i+1}} \frac{(d_1 + \qquad + d_{i+1})^r}{10^{s(N_{i+1}-1)}} \leq$$

$$\frac{d_{i+1}(d_1+\cdots+d_{i+1})^r}{10^{s(N_{i+1}-1)}}.$$

Thus,$\sum_{i\geq 0} \sum_{j=d_1+\cdots+d_i+1}^{d_1+\cdots+d_i+d_{i+1}} \frac{j^r}{a_j^s} \leq \sum_{i\geq 0} \frac{t(i+1)^r t^r}{10^{s(N_{i+1}-1)}} \leq$

$\frac{t^{r+1}}{10^{s(N_1-1)}} \sum_{i\geq 0} \frac{(i+1)^r}{10^{is}}.$

But, the last series converges so that $\sum_{n=1}^{\infty} \frac{n^r}{a_n^s}$ also converges. This is a contradiction of our assumption and the irrationality of x follows.

Suggested Reading

[1] **A McD Mercer, A note on some Irrational Decimal Fractions,** ***American Mathematical Monthly,*** **Vol. 101, pp.567-568, 1994.**