A Journey Along some Well-Known Curves

Utpal Mukhopadhyay

In mathematics, we encounter various types of curves. Some of these curves possess very interesting backgrounds. In this article, the historical backgrounds of some well-known curves have been discussed along with some other interesting features.

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Brachistochrone Problem

The Bernoulli family of Switzerland (originally from Holland) produced as many as eight mathematicians in three successive generations of whom at least three can be regarded among the greatest mathematicians of all times. In the June 1696 issue of *Acta eruditorum*, Johann Bernoulli I (AD 1667-1748), youngest son of Nicholas Bernoulli (AD 1623-1708), posed a typical mathematical problem before famous mathematicians of the world. The problem is – "What path a particle must follow so as to reach, starting from a fixed point and moving under gravity, to another fixed point in least time?" In Greek, 'Brachisto' means shortest and 'Chrone' means time. So, the above problem posed by Johann I is known in mathematical circles as 'Brachistochrone problem' (see [3]). Much earlier, Galileo (AD 1564-1642) gave his attention to this problem and concluded that the path would be a circle. Johann I gave six months time for solving the problem. Of the five solutions two came from Johann I and his elder brother Jacob I (AD 1654-1705) and two others came from Leibniz (AD 1646-1716) and Issac Newton (AD 1642-1727), two joint discoverers of differential calculus. The fifth solution came from L'Hospital (AD 1661-1704), writer of the first book *Analyse des infiniment petits* on calculus. It is worth mentioning here that the oldest scientific journal of the

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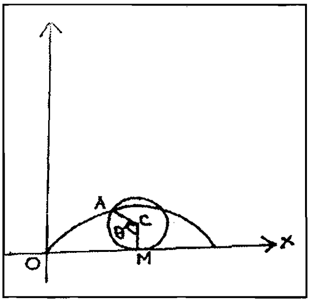


Figure 1. Cycloid.

world *Acta eruditorum* started publication in AD 1682 under the supervision of Leibniz. Anyway, all the five persons who correctly solved the problem concluded that the required path would be a cycloid, not a circle. So Galileo was wrong in predicting the path. In fact calculus, a formidable weapon of modern mathematics, was unknown to Galileo and probably for this reason it was not possible for a great scientist like Galileo to conclude correctly. The curve traced out by a point on the circumference of a circle rolling (without sliding) on a horizontal plane is a cycloid (*Figure* 1). The parametric equation of a cycloid is

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\} \quad (1)$$

where $\angle ACM = \theta$ (*Figure* 1).

Tautochrone Problem

In 1673, twenty three years before the Brachistochrone problem came to limelight, Christian Huygens (AD 1629-1695), another stalwart in science, directed his attention towards the solution of a mathematical problem. Huygens was seeking a curve such that a particle moving along the curve under gravity from a point on the curve to another point on it will always take the same time irrespective of the initial position from which the particle is released. This problem is known as the 'Tautochrone problem'. Huygens calculated that the curve would be a cycloid. When Johann I observed that the same curve (i.e. cycloid) was the solution of both the Brachistochrone and the Tautochrone problem, then he was so astonished as to remark – "But you will be petrified with astonishment when I say that exactly this same cycloid, the tautochrone of Huygens, is the brachistochrone we are seeking"

Cycloid's Cousins

Cycloid has two cousins – one is hypocycloid and another is epicycloid. Sometimes we find on a Kolkata pavement a man drawing various designs by rolling a toothed circular wheel. Actually, the circular wheel used by the man is nothing but a toy named 'spirograph'. This toy consists of a few circular toothed wheels accompanied by one or two bigger annular rings. The wheels are toothed along the outer periphery while the rings are toothed along both outer and inner circumferences. Each wheel has a few holes at different distances from its centre. An annular ring is placed on the paper and holding it firm, a wheel is placed on the paper touching the ring tangentially. Then, if the wheel is rotated by inserting the nib of a pen in one of the holes on the periphery, a curve appears on the paper. If the wheel is placed outside the ring, then the curve traced out is an epicycloid. On the other hand if the wheel is rotated by placing it inside, then a hypocycloid is generated. In Greek 'Epi' means over and 'Hypo' means under; hence the name. The size of the curve depends on the ratio of the radii of the wheel and the ring. If R is the radius of the ring and r that of the wheel, then the parametric equation of the hypocycloid becomes

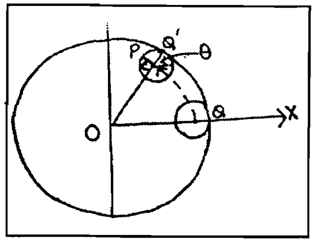


Figure 2. Generation of hypocycloid.

$$\left.\begin{aligned} x &= (R-r)\cos\theta + r\cos[(R-r)/r]\theta \\ y &= (R-r)\sin\theta - r\sin[(R-r)/r]\theta \end{aligned}\right\} \quad (2)$$

where θ is the amount of rotation of the centre of the wheel relative to the centre of the ring (*Figure* 2). The size of the hypocycloid depends on the value of R/r. If $R = 2r$, then from (2), we obtain

$$\left.\begin{aligned} x &= 2r\cos\theta \\ y &= 0 \end{aligned}\right\} \quad (3)$$

It can be seen from (3) that (since $y = 0$) the locus of a moving point P will lie along the x-axis, i.e. it will be a straight line. So by using two circles whose radii are in the ratio 2:1, one can draw a straight line! In the nineteenth century, the problem of converting a circular

In the nineteenth century, the problem of converting a circular motion into a rectilinear motion and vice versa was very important for designing steam engines because it was essential to convert the to and fro motion of the piston into a circular motion of the wheels.

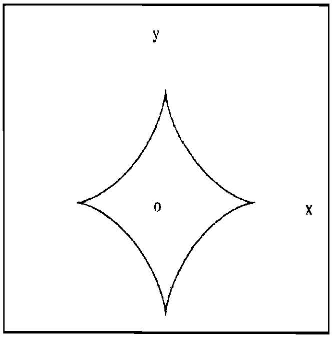


Figure 3. Astroid.

motion into a rectilinear motion and vice versa was very important for designing steam engines because it was essential to convert the to and fro motion of the piston into a circular motion of the wheels. One of the solutions of the problem was a hypocycloid with radii in the ratio 2:1.

Again, if $R = 4r$, then (2) reduces to

$$\left.\begin{aligned} x &= 3r\cos\theta + r\cos 3\theta \\ y &= 3r\sin\theta - r\sin 3\theta \end{aligned}\right\} \quad (4)$$

The Cartesian equation obtained from (4) is

$$x^{2/3} + y^{2/3} = R^{2/3}, \quad (5)$$

where $R = 4r$. Hypocycloid obtained in the form of (5) is called astroid (*Figure* 3). A funny thing about the astroid is that the sum of the intercepts on the axes made by any tangent of it is always constant. If a ladder is placed on a wall then the curve obtained from all possible positions of the centre of gravity of the ladder will take the shape of a hypocycloid. It can be shown using calculus that the perimeter of an astroid is $6R$. The area of the region bounded by an astroid is $\frac{3\pi R^2}{8}$, i.e. three-eighth of the area of a circle of radius R.

In 1725, Daniel Bernoulli (AD 1700-1782), eldest son of Johann Bernoulli I discovered a nice property of the hypocycloid. He showed that the hypocycloid generated by moving a circle of radius r on a ring of radius R is equivalent to the cycloid generated by moving a circle of radius $(R - r)$ on the same ring of radius R. This happens when the sum of the radii of the rotating circles is equal to the radius of the ring. This is known as Double Generation Theorem.

If a ladder is placed on a wall then the curve obtained from all possible positions of the centre of gravity of the ladder will take the shape of a hypocycloid.

Ancient Greeks played a leading role in developing the idea of an epicycle. Ideas of epicycloid originated for explaining the movements of planets in the sky. In order

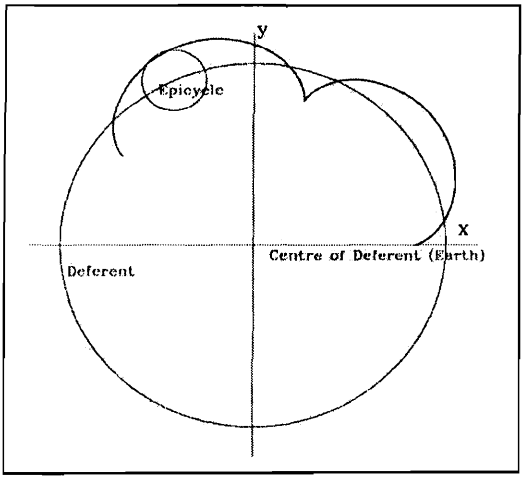


Figure 4. Epicycle and deferent.

to explain the retrograde motion (apparent backward motion of outer planets relative to earth) of planets in the sky, Greek astronomers invented epicycle theory. It should be kept in mind that heliocentric theory was unknown at that time and geocentric idea was prevailing all over the world. It was thought by Greeks, strong believers in aesthetics, that planetary motions could only be explained by circular motion as the circle was a symbol of a perfect curve to them. But they realised that simple circular motion was not sufficient to explain the movements of planets. For this reason, they hypothesized that planets move along a smaller circular path, called epicycle, and again the centres of those epicycles move along the circumference of a larger circle called deferent (*Figure* 4). Finally when the heliocentric idea of Copernicus came up in 1543, then the necessity of epicycle was over because it was realised that particular positions of planets relative to earth was responsible for retrograde motions of planets. In 1674, Danish astronomer Roemer (AD 1644-1710), while working on the mechanism of gears, investigated the cycloid family of curves.

The parametric equation of an epicycloid is

$$\left.\begin{aligned} x &= (R+r)\cos\theta - r\cos[(R+r)/r]\theta \\ y &= (R+r)\sin\theta - r\sin[(R+r)/r]\theta \end{aligned}\right\} \qquad (6)$$

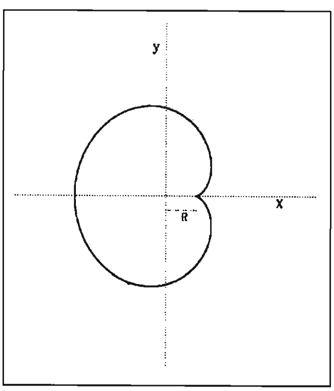
Box 1.

Maria Gaetana Agnesi (1718-1799 A.D.), daughter of Pietro who was a professor of mathematics in Bologna University in Italy, was born in Italy and spent almost all her life in that country. Pietro encouraged his daughter in learning science and for that purpose regularly invited scholars of various disciplines from all over Europe in front of whom Maria used to present research papers of her own on different subjects. Those subjects included philosophy, mechanics, chemistry, botany, zoology and metallurgy. At the young age of fourteen, Maria mastered the technique of solving difficult problems of analytical geometry and physics. At seventeen, she started writing her critical comments on *Traite des sectiones coniques* written by L'Hospital. From this period onwards she gradually shifted all her attentions to mathematics. Through the next ten years she wrote her main work *Instituzioni analytique ad uso della gioventu italiana* (Analytique institutions for the use of young Italian). This book was published in 1748 in two large volumes. The first part contains discussions on algebra and the second part deals with analysis. With the publication of the book, Agnesi's fame spread far and wide. John Colson, the Lucas Professor of Cambridge University, translated Agensi's book into English which was published in London in 1801. Within two years of the publication of her book, Pope Benedict XIV offered her professorship of mathematics in Bologna University in the year 1750. But, treating that post as honorary, she never joined that University. After the demise of her father in 1752, Agnesi withdrew herself from all research activities and devoted the rest of her life to religious and social activities. This talented mathematician passed away in Milan in 1799.

where R and r have their previous meanings.

The size of the epicycloid also depends on R/r. When $R = r$, then (6) reduces to

$$\left.\begin{aligned} x &= r(2\cos\theta - \cos 2\theta) \\ y &= r(2\sin\theta - \sin 2\theta) \end{aligned}\right\}. \tag{7}$$



Equation (7) represents a curve in the shape of a human heart called *cardioid* (*Figure* 5). The perimeter of a cardioid is $16R$ and its area is $6\pi R^2$.

Witch of Agnesi

The curve linked with the name of the famous mathematician Agnesi (see *Box* 1) is the 'Witch of Agnesi'.

Suppose the centre of a circle is at the point $(0, a)$ (*Figure* 6). A line passing through the origin cuts the circle at A and the line $y = 2a$ at B. The horizontal line

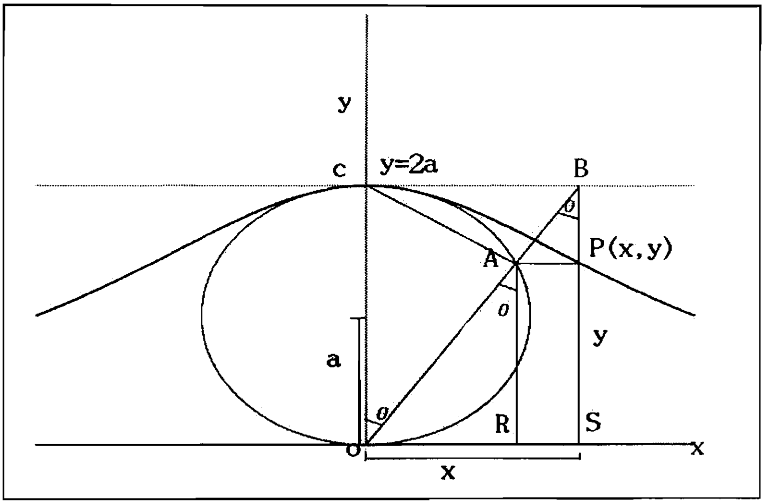


Figure 6. Witch of Agnesi.

through A and the vertical line through B intersect at P. Then the locus of P for all possible positions of OA is the curve 'Witch of Agnesi'. The parametric equation of this curve is

$$\left.\begin{aligned} x &= 2a\tan\theta \\ y &= 2a\cos^2\theta \end{aligned}\right\} \qquad (8)$$

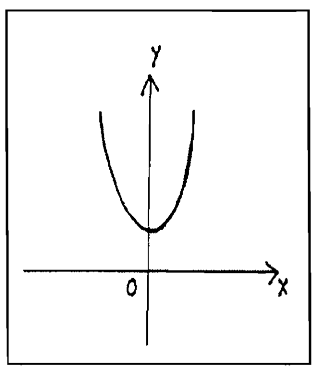
Though Agnesi's name is associated with this curve, some other mathematicians also paid their attention to it. Fermat (AD 1601-1665) was well aware of this curve and a professor of Pisa University christened it as 'Versiera' which originated from the Latin word 'Vertera' (meaning 'to turn'). But an Italian word with almost the same pronunciation is 'avversiera' which means witch. Perhaps in England the word 'Versiera' was translated as witch.

Why this curve attracted mathematicians' interest is not very clear. Perhaps, it was Agnesi who brought this curve into the limelight and hence her name was associated with this curve.

Catenary and the Gateway Arch

The Gateway Arch stands high on the bank of the river Missouri in St. Louis of Missouri Province. This monu-

Figure 7. Catenary.



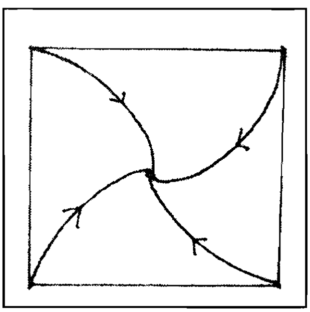
Various natural objects like conchshells, horns and trunks of elephants have a natural tendency to grow in the form of logarithmic spirals.

ment, designed by architect Eero Saarinen, was built in 1965 and its highest point is situated at a height of 630 ft. from the ground. Those who have some knowledge of mathematics may raise the question whether the arch looks like a known curve. Some of them may even think that the shape is that of a parabola. But no! Galileo also made the same mistake. "If a chain, whose ends are rigidly fixed, is allowed to hang under gravity then what will be the shape of the chain?" In the May, 1690 issue of *Acta eruditorum*, Jacob Bernoulli I threw this question to renowned mathematicians of the entire world. Much earlier than that, Galileo thought over the same problem and wrongly concluded that the curve would be a parabola. But, Galileo's mistake was pointed out in 1646 by none other than Huygens, then only a young lad of seventeen. He proved mathematically that the curve cannot be a parabola. But Galileo had already breathed his last four years before that. Anyway, in the June 1691 issue of *Acta eruditorum* three correct solutions of the problem were published which came from Huygens, Leibnitz and Johann Bernoulli I. All of them concluded that the curve would be a catenary (*Figure* 7). Jacob Bernoulli I himself could not solve the problem. The word 'catenary' originated from a Greek word 'Catena' which means chain. The Gateway Arch was built in the form of an inverted catenary. The Cartesian equation of a catenary is

$$y = \frac{e^{ax} + e^{-ax}}{2},$$

where 'a' is a constant whose value depends on per unit mass and tension of the chain. Though a catenary looks like a parabola, it is not exactly a parabola.

Figure 8. Paths of bugs.



Spira Mirabilis

Suppose, four bed bugs are sitting at four corners of a square and with the blinking of a signal (it should be mentioned here that the bugs are intelligent enough to understand the signal) they start moving towards the adjacent one. Then what will be the paths of the bugs? This is a mathematical problem known as 'Four Bug Problem'. Mathematicians have shown that the paths of the bugs will be logarithmic spirals which meet at the centre of the square (*Figure* 8). A logarithmic spiral is that curve which was nicknamed 'Spira mirabilis' (marvellous spiral) by Jacob Bernoulli I. Observing various peculiar properties (see [3]) of that curve, Jacob I wished that a logarithmic spiral be engraved on his tombstone with the words 'Eadem mutata resurgo' (Though changed, I shall arise the same) written under it. The polar equation of a logarithmic spiral is $r = e^{a\theta}$.

Perhaps no other curve caught so much attention of scientists, naturalists and artists like the logarithmic spiral. The British naturalist D'Arcy W Thomson (AD 1860-1948) in his book *On Growth and Form* has discussed about the natural tendency of growth of various natural objects like conchshell, horn, trunk of an elephant, etc. in the form of a logarithmic spiral. In the early years of the twentieth century, as a result of resurrection of ideas about the relationship between Greek art and mathematics, new light was thrown on the logarithmic spiral. In 1914, Sir Theodore Andrea Cook, in his book *The Curves of Life*, discussed about the role of logarithmic spiral in nature as well as in the field of art. In this way, the logarithmic spiral has tied up the people from various disciplines into one knot! Jacob Bernoulli I had aptly given the name 'Spira mirabilis' to the logarithmic spiral.

Perhaps through the above discussion, it has been possible to reveal that mathematics is not a mere collection of formulae and theorems, but contains some interesting features also. If the historical background of the topics taught is also touched upon in brief in the classroom, then probably the teaching and learning processes will be more fruitful and will not be turned into a drudgery.

Suggested Reading

[1] Eli Maor, *e, The Story of a Number*, Universities Press, 1999.

[2] Eli Maor, *Trigonometric Delights*, Universities Press, 2000.

[3] Utpal Mukhopadhyay, Bernoulli Brothers, Jacob I and Johann I: A Pair of Giant Mathematicians; *Resonance*, Vol. 6, No. 10, Oct. 2001.

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