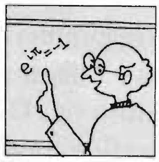
Classroom



In this section of *Resonance*, we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.

Proving a Result in Combinatorics using Equations

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Introduction

We shall prove a result in combinatorics using the equation

$$x_1 + x_2 + \quad + x_m = n \tag{1}$$

First we compute the number of non-negative integral solutions to (1) (see [1]). Let us take a simpler illustration of the above equation. Consider the equation $x_1 + x_2 = 6$.

x_1	x_2	Binary Sequence
6	0	1 1 1 1 1 1 0
0	6	0 1 1 1 1 1 1
5	1	1 1 1 1 1 0 1
1	5	1 0 1 1 1 1 1
4	2	1 1 1 1 0 1 1
2	4	1 1 0 1 1 1 1
3	3	1 1 1 0 1 1 1

The third column of the table lists all the binary sequences containing 6 ones and 1 zero; each uniquely

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The number of non-negative integral solutions of the equation $x_1+x_2+ \ldots +x_m = n$ is $\binom{n+m-1}{m-1}$

corresponds to a solution of $x_1 + x_2 = 6$. The number of ones to the left of the zero is x_1, and the number of ones to the right of the zero is x_2. So the solutions are in 1–1 correspondence with the permutations of 6 ones and 1 zero. This number is $\frac{7!}{6!\,1!} = 7$. So there are precisely 7 solutions. Generalizing, we see that the number of non-negative integer solutions of (1) is $\binom{n+m-1}{m-1}$.

Next we will find the number of positive integer solutions (see [2]) to equation (1). Consider a line segment of length n. Any solution of (1) in positive integers corresponds to a decomposition of this segment into m pieces, their lengths being positive integers. The $m-1$ end-points of these pieces must be chosen from among the $n-1$ points at distances 1, 2, 3, $\ldots$, $n-1$ from one end-point. This can be done in $\binom{n-1}{m-1}$ ways which gives the number of positive integral solutions to the equation.

Proving an Identity in Combinatorics using Equations

We shall use the above to prove that for $n \geq m$ and $n \geq 1$, the sum

$$\binom{n-1}{0}\cdot\binom{m}{1}+\binom{n-1}{1}\cdot\binom{m}{2}+\binom{n-1}{2}\cdot\binom{m}{3}+\cdots+\binom{n-1}{m-1}\cdot\binom{m}{m} \text{ equals } \binom{n+m-1}{m-1}.$$

Proof. We shall prove the result by counting the number of non-negative integral solutions to

$$x_1 + x_2 + \cdots + x_m = n$$

in another way. As just shown, the number of non-negative integral solutions to this equation is $\binom{n+m-1}{m-1}$. These solutions include the positive integral solutions and also the solutions in which one or more of the x's is zero. We count the solutions by considering the various possibilities.

- *Suppose that $n-1$ of the x's are* 0. Then one of the x's is n. Suppose that $x_1 = n$. The number of positive integral solutions to this equation is $\binom{n-1}{0} = 1$. Choosing which of the x's is n can be done in $\binom{m}{1}$ ways. So the number of non-negative integral solutions in this case is $\binom{n-1}{0}\binom{m}{1}$.

- *Suppose that $n-2$ of the x's are* 0. The number of positive integral solutions to $x_1 + x_2 = n$ is $\binom{n-1}{1}$, and the number of ways of choosing two of the x's is $\binom{m}{2}$. Hence the contribution is $\binom{n-1}{1}\binom{m}{2}$.

- *Suppose that $n-3$ of the x's are* 0. The number of positive integral solutions to $x_1 + x_2 + x_3 = n$ is $\binom{n-2}{1}$, and the number of ways of choosing three of the x's is $\binom{m}{3}$. Hence the contribution is $\binom{n-2}{1}\binom{m}{3}$.

- Continuing, the last case to consider is where none of the x's is 0. The number of positive integral solutions to $x_1 + x_2 + \quad + x_m = n$ is $\binom{n-1}{m-1}$. The contribution in this case is $\binom{n-1}{m-1}\binom{m}{m}$.

Adding all these contributions, we get the stated result. Hence proved.

We will illustrate this with an example. Consider the equation $x_1 + x_2 + x_3 = 6$. Here $n = 6$ and $m = 3$. Therefore the number of non-negative integer solutions is $\binom{8}{2} = 28$. Let us now consider the various possibilities.

- $x_1 = 6$: we have $\binom{5}{0}\ \binom{3}{1} = 3$.
- $x_1 + x_2 = 6$: we have $\binom{5}{1}\ \binom{3}{2} = 15$.
- $x_1 + x_2 + x_3 = 6$: we have $\binom{5}{2}\ \binom{3}{3} = 10$.

Adding, we get $3 + 15 + 10 = 28$ which is equal to the number of non-negative integral solutions.

Suggested Reading

[1] **V Krishnamurthy *et al*, *Challenge and Thrill of Pre-College Mathematics*, New Age International Publishers, New Delhi, 1996.**

[2] **A M Yaglom and I M Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Vol.1, Dover Publications, Inc, New York, 1987.**