# Beyond Brownian Motion: A Levy Flight in Magic Boots

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**In recent times, it has become quite common in the scientific community to observe molecular aggregates which travel at times like Tom Thumb wearing his magic boots. This requires an understanding of theories going beyond the familiar Brownian Motion (BM).**

## Brownian Motion

Brownian motion is the simplest of random walks. It is a common phenomenon in a broad spectrum of systems in Chemistry, Physics and Biology. Fundamental dynamical processes such as molecular transport in liquids, atomic and molecular diffusion on surfaces, motion of microorganisms as well as many other examples have all been described in terms of BM. To cite a couple of examples from our daily activities, BM explains why a spoonful of sugar sweetens a whole cup of coffee and why the perfume worn by one person permeates a closed room.

The age-old but vivid picture generally evoked to describe BM is that of a drunkard wandering from lamp post to lamp post and in his inebriated state, forgetting each time the place he came from. He chooses his new direction at random. BM addresses the following questions: What distance does he cover in time $t$? How much time elapses before he reaches his destination for the first time? This completely erratic movement is mimicked by very small particles, small enough to be affected by unending collisions with the surrounding atoms, which hit them from all sides. This is called Brownian motion after the Scottish botanist, R Brown, who studied the motion of pollen grains in 1827. Everything is subject

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to thermal fluctuations and molecules, macromolecules, viruses, particles and other components of the natural world are all in eternal motion with random collisions due to thermal energy.

BM presents universal characteristics common to all random movements. A detailed study of BM was made by Einstein in 1905. The important result that emerged was that the distance of the Brownian particle from its starting point increases with the square root of time and not linearly. Since this is a random phenomenon, the probability of a particle being at distance $d$ from its starting point after time $t$ is represented by a 'Gaussian'- a bell shaped curve which very rapidly drops to zero (*Figure* 1) centred around the origin and having a width proportional to $\sqrt{t}$. The width of the bell is a measure of the distance beyond which there is little probability of finding the particle. A striking aspect of this result is its universality. Whatever the microscopic details (nature of the particle, its surroundings, temperature etc.) are, one will always find after a sufficient time this Gaussian law of width $\sqrt{t}$. In other words, our drunkard may not be completely drunk, the lamp posts may not be regularly distributed, but his uncertain movement will always be diffusive, i.e., possessing the characteristics of *Figure*1.

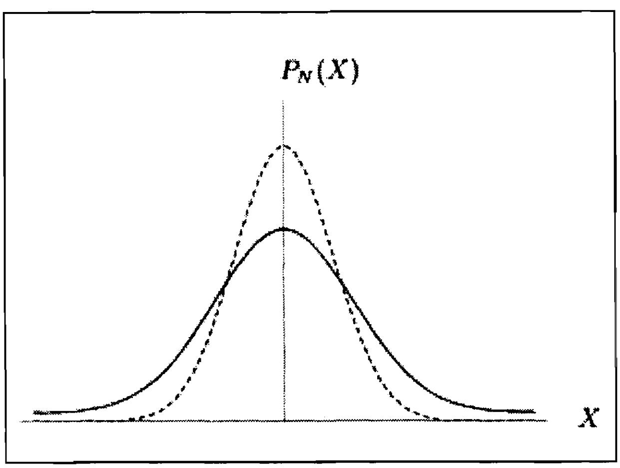


***Figure 1. According to the Central Limit theorem, the probability $P_N(X)$ of the total displacement X, which is the sum of a large number of random displacements has a standard form called the Gaussian (the dotted curve). As X goes far from the mean value, the probability rapidly drops to zero. Though this law of probability is almost universal, there are some exceptions. In the Levy law, the probability curve has large tails (the solid curve). Atypical values become relatively probable thus rendering it impossible to determine a mean value or a typical deviation.***

Deviation from the Gaussian curve would require a situation involving very strong fluctuations, where the probability of observing atypical events is not too small.

## The Central Limit Theorem (CLT)

The universality of this diffusion law can be understood, when one recalls the 'central limit theorem' (CLT), which describes the behaviour of sums of random variables. If $\{x_1, x_2, \quad, x_N\}$ is a series of $N$ random numbers, their sum $X = x_1 + x_2 . \; . + x_N$ is for very large $N$, a variable, whose law of probability is a Gaussian. Consider the following example: Suppose $x_i$ represents the result of the throw of an $i$th die. This result is one of the whole numbers from 1 to 6, each having the same probability of occurrence of 1/6. For $N$ dice, the variable $X$ will represent the sum of the $N$ results obtained, which will be between $N$ and $6N$ For this case the CLT indicates that for large values of $N$, the probabilities $P_N(X)$ of the various possible values of $X$ will be distributed according to a Gaussian curve (*Figure* 1).

The CLT is valid under very general conditions. Whatever the law of probability of $x_i$ values and their mutual correlations, the probability distribution of $X$ always remains a Gaussian curve, whose width increases as $\sqrt{N}$ Thus the Gaussian acts as a sort of 'black hole' of statistics and attracts and controls a vast majority of sums of random variables. In general, the total displacement of a Brownian particle is the sum $X$ of the small, elementary random displacements $x_i$ resulting from collisions with the surrounding molecules. The Gaussian law predicted by the CLT for this problem is almost always observed experimentally.

## Deviations from CLT and Levy Flights

The question is: Can there be instances when the central limit theorem is evaded and one obtains an 'abnormal' or a non-Gaussian diffusion? This would require a situation involving very strong fluctuations, where the probability of observing atypical events is not too small. For example, if the cup of coffee is energetically stirred, turbulence may occur and the sugar may be dispersed more rapidly, and the particle motion can no longer be described in terms of a Gaussian distribution. The CLT in its usual form does not apply if the probability $p(x)$ of obtaining a given value of $x$ decreases less rapidly than $1/x^3$ for large x.

The generalisation of the CLT to these cases is due to Paul Levy (1935). His work, which began the study of what are now called Levy flights, was seminal in the study of non-Brownian enhanced diffusion. The consequence of these rebel fluctuations is two-fold. The sum $X$ increases faster than the square root of the number of terms it contains, and the distribution obtained is no longer a Gaussian, but obeys a law called the stable or Levy law (*Figure* 1). The Levy sums are exceptional in that they are rigorously 'hierarchical' A very small number of terms dominate all others and the contribution of the latter to the sum $X$ is negligible. Hence this sum reflects mainly the value of the largest terms. On the other hand, Gaussian sums are democratic, with each term contributing significantly to the final result.

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Over the last twenty five years, physicists have been investigating Levy laws. It is rare that a mathematical result is not eventually used to describe a physical phenomenon. Gradually the atypical fluctuations have been observed very frequently, where the determinant event is often the rare event. The pioneering work of Benoit Mandelbrot showed that such Levy distributions are common in nature for example, they describe the way in which albatrosses search for food, the fluctuations in share prices, and the rhythms of the heart. It was Mandelbrot who coined the term 'Levy Flight' for the natural generalisation of BM to situations of strong fluctuations, i.e., for super diffusive displacements where the motion is more rapid than that given by the usual diffusion law.

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## Levy Flights and Micelle Dynamics

Levy flights were never observed experimentally till 1990. They were discovered by chance by A Ott and coworkers of the Ecole Normale Superieure in Paris while studying the diffusion of fluorescent molecules within an assembly of giant cetyl trimethyl ammonium bromide micelles in salted water. These micelles are long, flexible cylinders, of about $50A^{\circ}$ in diameter, made of surface active molecules and salts. Seen from a distance, these micelles resemble polymers, but unlike polymers, they split and recombine continuously and at random. Some of these cylinders are extremely long while others are much shorter. It is obvious that the movement of each cylinder is hindered by the presence of others. To move forward, it must sneak out like a snake. But, the longer its tail, the slower is its movement. A micelle which is ten times shorter than another will cover in a given time, a distance ten times greater. The technique of fluorescence recovery after fringe-pattern photobleaching (FRAP) was used to study the fluorescent molecules attached to these micelles.

Every time a micelle splits and recombines with another one, the fluorescent molecule is found on a micelle whose size and hence the mobility is changed. In the course of time, the fluorescent molecule borrows a series of vehicles with varying performances. On estimating the total distance covered, it is surprising to note that it is the shortest micelle which made the fluorescent molecule travel the major part of this distance in one strike, the total contribution of other micelles being comparatively small. This may be compared to the distance covered by a person travelling successively on foot, on a horse, by car and by air. The major contribution to the distance covered is always by the most efficient mode of transport. One can make this description more rigorous and show that the movement of the fluorescent molecules is exactly a Levy flight (*Figure* 2). The experiment shows

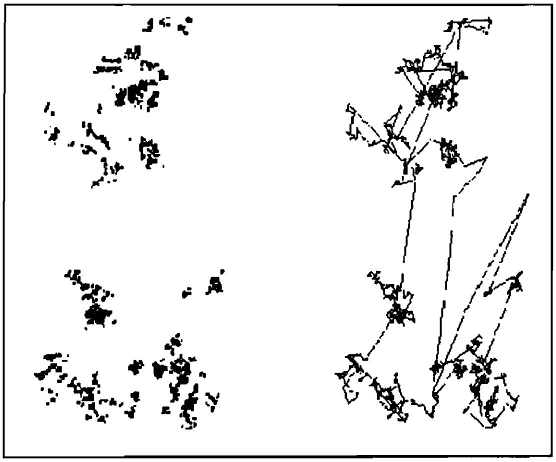

*Figure 2. The trajectory of a particle executing a 'Levy flight'. Each line corresponds to a leap, but the leaps are arranged in a hierarchical manner. The long leaps are rare but they are the ones that contribute to the major distance covered by the particle. Also, the set of points where the trajectory changes its direction is a fractal set.*

a super diffusive displacement that is more rapid than the usual diffusion law as well as the law of non-Gaussian probability, described by a Levy law. The establishment of Levy flights in these systems of giant micelles confirms the theoretical models proposed to explain their properties.

## Other Levy Manifestations

In 1993, C-K Peng and colleagues from Boston University, Harvard Medical School and the Bethesda National Institutes of health analysed the time intervals between heartbeats. They found that the erratic patterns observed in the heartbeats of healthy subjects can be described by a Levy distribution while data from patients with severe heart failure are much closer to a Gaussian distribution. This difference could help in understanding the detailed physiological processes that control heart activity. They suggested that these results could arise from a nonlinear competition between branches of the involuntary nervous system. In the same year, Harry Swinney and colleagues at the University of Texas studied the flow of a liquid in a rotating vessel. The experimental setup essentially generates fluid flow in two dimensions. Swinney's group found that vortices, a signature of turbulence, appeared at various places in the liquid. Tracer particles were followed for long peri-

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ods of time and were found to alternate between staying in a particular vortex and flying towards a neighbouring one. The flights between vortices follow a Levy distribution.

In 1995 a group led by Paulo Murilo Oliveira and Thadeu Penna of Niteroi University in Brazil used both experimental and numerical methods to study a leaking tap. They found that the time intervals between drops fluctuate with a Levy distribution. Also in 1995 Rosario Mantegna and Gene Stanley of Boston University showed that financial markets also can follow Levy distributions. Their study could provide a framework for developing economic models of share prices.

The list of Levy manifestations in nature does not stop there. Other examples include subrecoil laser cooling studied in 1994 by F Bardou and colleagues, also at the Ecole normale Superieure in Paris. In 1996, researchers from the Boston University and the British Antarctic Survey found that even the wandering albatrosses live their lives by a Levy distribution. When looking for food, these seabirds fly for long distances, then forage in a small area before flying off again. Scientists are now investigating as to whether the foraging behaviour of other species, such as ants and bees also follow Levy distributions.

### Levy Flights and Fractals

Levy posed the question: When does the probability for the sum of $N$ steps $P_N(X)$, where $X = x_1+x_2+. .+x_N$, have the same distribution (up to a scale factor) as the probability $p(x)$ for the individual steps? This is basically the question of fractals: When does the whole look like its parts? How to characterise the limit distribution of the sum of independent random variables when the distribution, $p(x)$ is 'broad'(that is, decreases more slowly than $x^{-3}$ for large $x$)? The pertinent result from

Levy's analysis is that in the Fourier space,

$$\widetilde{P}_N(k) = \exp\left(-N|k|^{\gamma}\right) \tag{1}$$

Here $\gamma$ can take values between 0 and 2. This corresponds to $p(x) \sim x^{-1-\gamma}$ for $x \to \infty$. In other words, the variance, $\langle |x|^2 \rangle$ diverges for $\gamma < 2$. This means that there is no characteristic size for the RW jumps, except in the Gaussian case of $\gamma = 2$. It is just this absence of a characteristic scale that makes Levy flights scale invariant fractals. In other words, Levy flights represent a random walker who visits a disconnected set of self-similar clusters of fractal dimension $\gamma$ for which the mean squared displacement $\langle |x|^2 \rangle$ diverges.

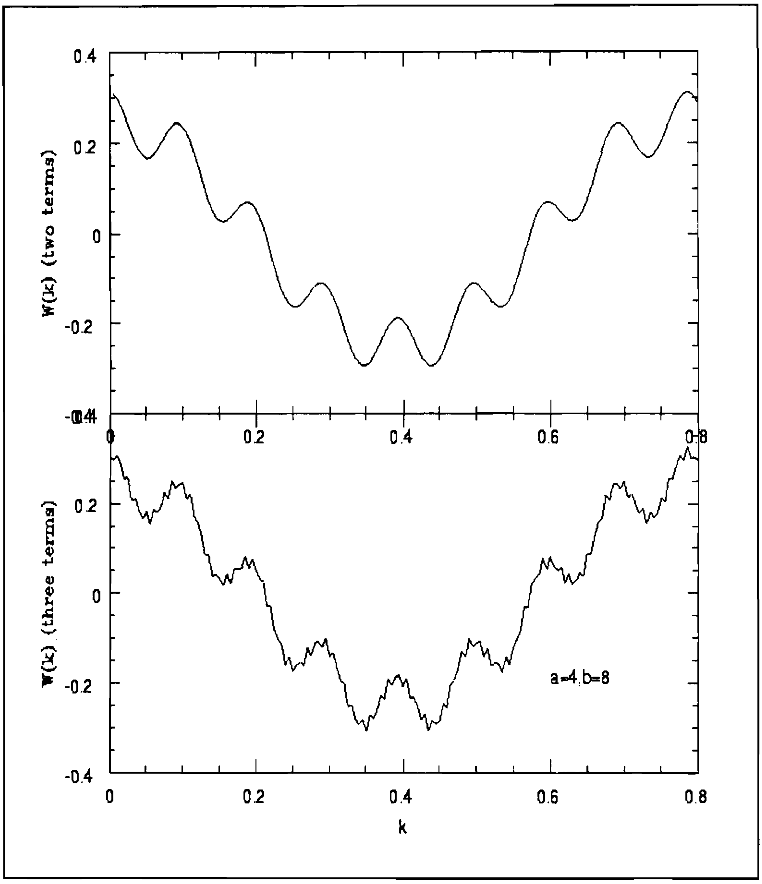
In Weierstrass random walk, there are jumps of length 1, *b*, $b^2$, $b^3$... But whenever the length of the jump increases by an order of magnitude (in base *b*), its probability of occurrence decreases by an order of magnitude (in base $\lambda$).

Let us consider a particular case called the Weierstrass random walk to illustrate the self-similarity of Levy flights For a one-dimensional random walk, the probability for a jump of displacement $x$ be

$$p(x) = \frac{\lambda - 1}{2\lambda} \sum_{j=0}^{\infty} \lambda^{-j} \left[\delta\left(x, +b^j\right) + \delta\left(x, -b^j\right)\right] \tag{2}$$

Here, $\delta(x, y)$ is the Kronecker's delta function; it equals one for $x = y$ and zero otherwise. $b > \lambda > 1$ are parameters that characterize the distribution. This is called a Weierstrass random walk. This distribution function allows jumps of length $1, b, b^2, b^3,$ But whenever the length of the jump increases by an order of magnitude (in base $b$), its probability of occurrence decreases by an order of magnitude (in base $\lambda$). Bernoulli scaling is apparent here an order of magnitude more but an order of magnitude less often which is the basis for the concept of fractal time. Thus there is a cluster of $\lambda$ jumps roughly of length 1 before there is a jump of length $b$. About $\lambda$ such clusters separated by lengths of order $b$ occur before one sees a jump of order $b^2$. It continues in this fashion, a hierarchy of clusters within clusters. The

*Figure 3. The figure shows how each new term in the Weierstrass function $W(k)=\Sigma_{n=0}^{\infty} a^{-n} \cos(b^n k)$ an order of magnitude more wiggles in the previous wiggle. The upper panel shows the contribution of first two terms and the bottom panel shows the effect of addition of the third term (for a=4, b=8).*




Fourier transform of $p(x)$ is

$$\widetilde{p}(k) = \frac{\lambda - 1}{\lambda} \sum_{j=0}^{\infty} \lambda^{-j} \cos\left(b^j k\right) \qquad (3)$$

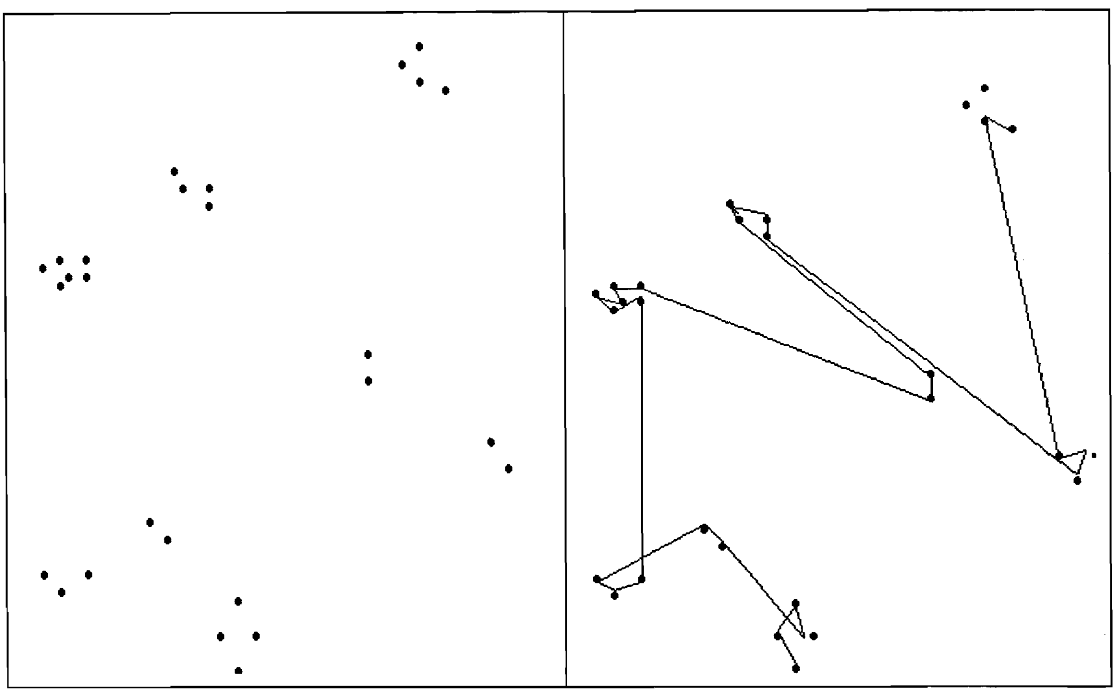
This is the famous Weierstrass function which is everywhere continuous but nowhere differentiable when $b > \lambda$. Each new term in the sum places an order of magnitude more wiggles (in base $b$) on a previous wiggle (*Figure* 3). Each new wiggle is reduced in amplitude by an order of magnitude (in base $\lambda$). The nondifferentiable Weierstrass curve has the fractal dimension, $\ln \lambda / \ln b$, which is also the fractal dimension of the Weierstrass walk.

## A Levy Walk or a Levy Drive

In spite of the elegance of Levy flights, the infinite moments of the distribution proved a stumbling block for any meaningful use as a mathematical device to tackle physical problems. The divergence of the moments can be tamed by associating a velocity with each flight segment. This prompts one to ask how far a Levy walker has wandered from his starting point in time $t$, rather than what the mean squared length of a completed jump is. While the first query elicits as reply, a well-behaved time-dependent moment of the probability distribution, the answer to the second is infinity. This random walk with a velocity is called a Levy walk or a Levy drive to distinguish it from a Levy flight where the walker visits only the endpoints of a jump and the notion of velocity does not arise (*Figures* 4 and 5). Thus for a single jump in a Levy flight,the walker is only at the starting point and at the end point and never in between. For a jump in a Levy drive, the walker follows a continuous

***Figure 4 (left). The first few points visited by a Levy flight. A fractal set of points (clusters within clusters within clusters, etc.) is eventually visited.***

***Figure 5 (right). The set of points visited by the corresponding Levy walk. It includes the same set of points as the Levy flight, plus the trail connecting these points. The Levy flight points are turning points of the Levy walk. Different velocities can be associated with different path lengths.***



trajectory from its starting to its end points and hence a finite time is needed to complete the drive. The sites visited by the Levy flight are the turning points of the Levy walk. The mean square displacement for a jump in a Levy flight is infinite and not useful. For a Levy drive the random walker moves with a finite velocity and hence its mean square displacement is never infinite but it is a time dependent quantity.

For a Levy drive, one introduces velocity through a coupled spatial and temporal probability $\Psi(r,t)$ for a random walker to undergo a displacement $r$ in time $t$.

$$\Psi(r,t) = \psi(t|r)p(r). \tag{4}$$

$p(r)$ is just the probability function for a single jump. $p(r) \sim r^{-1-\gamma}$ and $\psi(t|r)$ is the conditional probability that the jump takes a time $t$ given its length is $r$. For simplicity,

$$\psi(t|r) = \delta\left(t - \frac{|r|}{v(r)}\right) \tag{5}$$

This ensures that $r = vt$. An important feature of Levy drive approach is that it provides a method for simulating trajectories of turbulent particles. Turbulent motion is a case of enhanced diffusion wherein $\langle |x|^2 \rangle \sim t^3$, $x$ being the separation between two particles in a turbulent flow.

## Concluding Remarks

Thus a study of Levy flights and its modifications has been driven by the need to go beyond the regular Brownian formulation and describe the anomalous behaviour, in particular, the enhanced diffusion. Mathematical curiosities at the beginning, Levy laws have now become indispensable in investigations of the anomalous transport properties.

## Suggested Reading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