Planetary Orbits as Simple Harmonic Motion

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I like to ask my undergraduate students the following question: Consider a planet in a perfectly circular orbit around the Sun. Now take a celestial hammer and give it a slight radial knock. What happens? The answer, of course, is that the planet oscillates radially as it goes around the Sun. Radial oscillation combines with circular motion to give a plausible planetary orbit. So long as the eccentricity of the orbit is small, the radial motion is harmonic and determining the orbit is simpler than solving the general problem of planetary orbits. This approach is perhaps more insightful: it allows us to understand intuitively why a planetary orbit is closed, and why it is stable, and, with a little generalization, to determine the effect on a planetary orbit of the oblateness of the Sun and of corrections due to the general theory of relativity.

1. Why Planetary Orbits are Closed

Consider the gravitational potential of the Sun (mass M). The equation of motion of a planet at position $\mathbf{r}$ in such a potential is

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{GM}{r^3}\mathbf{r}. \tag{1}$$

(We assume that the mass of the planet is negligible compared to the mass of the Sun, so that the centre of mass coincides with the centre of the Sun and the reduced mass is equal to the mass of the planet.)

Since conservation of angular momentum confines the orbit to a plane, we can use polar coordinates (r, ϕ) to rewrite this equation as

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2} \tag{2}$$

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and

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = 0. \tag{3}$$

The second equation allows us to define a conserved quantity $l = r^2\dot{\phi}$, which is just the magnitude of the angular momentum (per unit mass). Thus we can write

$$\ddot{r} = -\frac{GM}{r^2} + \frac{l^2}{r^3}. \tag{4}$$

Now let us assume that our particle is in an orbit that is only slightly non-circular, i.e.

$$r = r_o + \delta, \tag{5}$$

where $\delta/r_o \ll 1$. Inserting $r_o + \delta$ in place of r and expanding to first order in δ/r_o, we get

$$\ddot{\delta} = -\frac{GM}{r_o^2} + \frac{l^2}{r_o^3} + \left(\frac{2GM}{r_o^3} - \frac{3l^2}{r_o^4}\right)\delta. \tag{6}$$

From (4) we know that for a circular orbit $l^2/r^3 - GM/r^2 = 0$. Since the orbit we are considering is only slightly non-circular, we may think of it as a superposition of a circular orbit (of radius r_o) and a small radial variation. This tells us that in (6) the first two terms cancel each other out, leaving

$$\ddot{\delta} = -\left(\frac{3l^2}{r_o^4} - \frac{2GM}{r_o^3}\right)\delta. \tag{7}$$

This is the equation of a simple harmonic oscillator with a frequency κ given by

$$\kappa^2 = (\frac{3l^2}{r_o^4} - \frac{2GM}{r_o^3}). \tag{8}$$

We may therefore write

$$\delta = \delta_o \cos \kappa t \tag{9}$$

Since the angular frequency of rotation $\Omega = \sqrt{l/r_o^2} = \sqrt{GM/r_o^3}$, we find that $\kappa = \Omega$, i.e. exactly one radial oscillation is completed in the time it takes for the planet to go once around the Sun. This implies that the orbit is closed. (There is another physically realizable gravitational potential in which the orbits are closed; can you determine what that is?)

The orbit is closed if $K = \Omega$.

2. Comparison with the Exact Orbit

When $\kappa = \Omega$, the azimuthal angle $\phi = \Omega t = \kappa t$, and so

$$r = r_o\left(1 + \frac{\delta_o}{r_o}\cos\phi\right) \tag{10}$$

For comparison, the exact answer is

$$r = \frac{r_o}{1 - \epsilon\cos\phi}. \tag{11}$$

Expanding this in a power series, we get

$$r = r_o(1 + \epsilon\cos\phi + \epsilon^2\cos^2\phi + ...). \tag{12}$$

The orbit of the Earth has an eccentricity of 0.0167; the approximate orbit calculated by the method we have outlined is correct to within a fraction of a percent.

Let us see what else we can learn from this approach to planetary orbits.

3. The Inverse-Square Law

So far we have *assumed* that gravity is an inverse-square force. We could turn the problem around and ask ourselves if any other force law is compatible with the patterns that Kepler discovered in Tycho Brahe's observations. The first, we all know, is that the planets move in ellipses with the Sun at one focus.

We may rewrite (8) as

$$\kappa^2 = \frac{d^2\Phi}{dr^2} + \frac{3l^2}{r^4}. \tag{13}$$

For a circular orbit to be stable any small deformation must be opposed by the system.

We have written r for r_o because we want to know how κ varies from orbit to orbit as we move out from the centre. The potential being spherically symmetric is characterized at any r entirely by the angular frequency of the circular orbit at that radius:

$$\Omega^2(r) = \frac{1}{r}\frac{d\Phi}{dr} = \frac{l^2}{r^4}. \tag{14}$$

Using this we can write

$$\kappa^2(r) = r\frac{d\Omega^2}{dr} + 4\Omega^2 \tag{15}$$

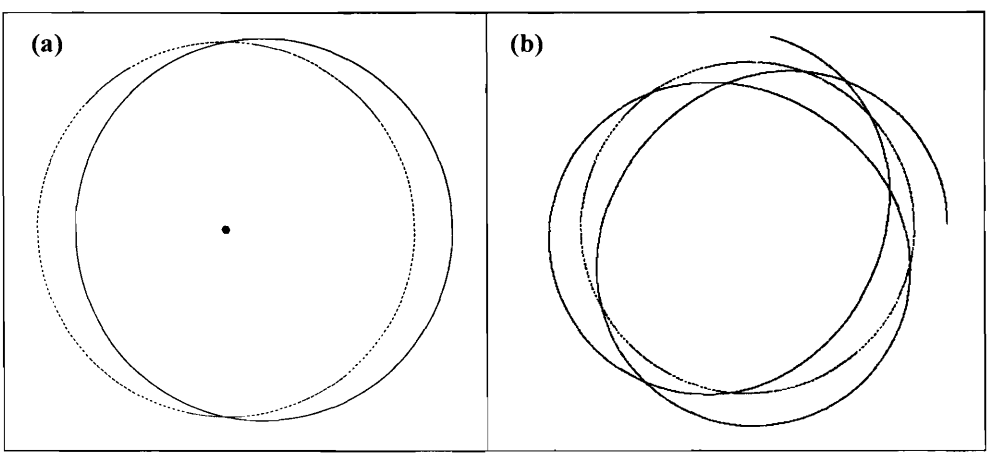
For a potential that varies as r^α, it is not difficult to show that $\kappa^2 = (2+\alpha)\Omega^2$. Only for the Kepler potential $(\Phi \propto 1/r)$ – which corresponds to an inverse-square law for the force – do the orbits satisfy Kepler's first law. (*Figures* 1a and 1b show orbits when $\kappa = \Omega$ and when $\kappa \neq \Omega$.)

Figure 1a. An epicyclic orbit for $\kappa = \Omega$. The dashed line shows the basic circular orbit.

Figure 1b. An epicyclic orbit for $\kappa = 1.3\ \Omega$.

4. Stability of Circular Orbits

Notice that when $\alpha < -2$, κ becomes imaginary. What does this mean? For a circular orbit to be stable any small deformation must be opposed by the system. This corresponds to a real positive frequency of oscillation for small perturbations. On the other hand, an imaginary



frequency of oscillation indicates that a small deformation becomes a runaway process. The most visual and intuitive way to understand this is to look at the effective potential of a Kepler orbit. Notice that (4) can be written as

$$\ddot{r} = -\frac{d\Phi_{eff}}{dr}, \tag{16}$$

where

$$\Phi_{eff} = -\frac{GM}{r} + \frac{l^2}{2r^2}. \tag{17}$$

A stable circular orbit corresponds to a minimum of Φ_{eff}. An elliptical orbit seen as a perturbation of a circular orbit corresponds to an oscillation between two points on the walls of the potential near the minimum. In *Figure* 2 are shown two graphs of Φ_{eff} versus r, one for the $1/r$ potential and another for the borderline $1/r^2$ potential. Notice that in the latter there is no minimum, and therefore no stable circular orbit.

5. Precession of the Perihelion of Mercury

We have seen that in the $1/r$ potential of a spherically symmetric mass distribution we get closed Keplerian orbits. But we know that the Sun is not quite spherically

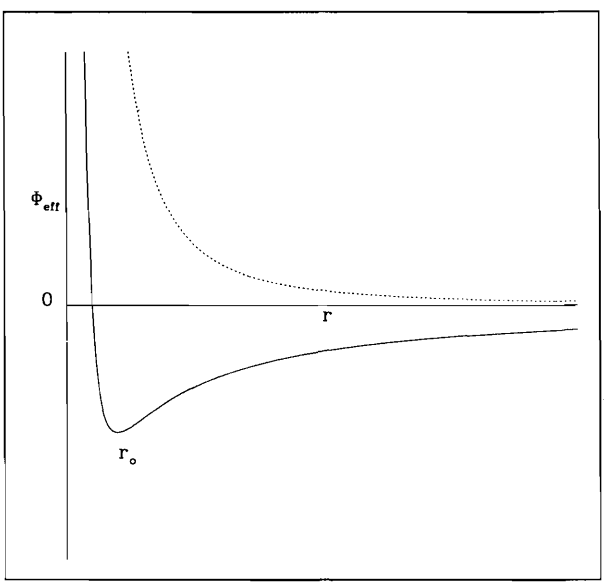


Figure 2. The effective potential when $\Phi \propto 1/r$ (solid line) and when $\Phi \propto 1/r^2$ (dashed line).

The Sun is somewhat flattened at the equator due to rotation.

symmetric; because of rotation it is somewhat flattened at the equator. In addition, we now know that Newton's theory of gravity must be modified in light of the general theory of relativity. Let us try to undestand how these two factors affect a planet's orbit.

5.1. *Oblateness of the Sun*

You learn in electricity and magnetism that the electrostatic potential due to an arbitrary distribution of charge can be expressed in terms of multipoles. Similarly, the gravitational potential due to an arbitrary distribution of mass can be expressed as the sum of a monopole term, a quadrupole term, and higher-order terms. (Question: Why is there no dipole term for a gravitational potential?) A spherically symmetric mass distribution has only a monopole. A uniform disk of radius a has a potential that looks like

$$\Phi(r,\theta) = -\frac{GM}{r}\left(P_0(\cos\theta) - \frac{1}{4}\frac{a^2}{r^2}P_2(\cos\theta) + \quad\right), \tag{18}$$

where θ is the angle with respect to the axis of symmetry, and the $P_n(\cos\theta)$ are the Legendre polynomials. (Those of you who know how to solve Laplace's equation can try working this out.) It is clear, therefore, that an oblate Sun will have a potential that looks like

$$\Phi(r,\theta) = -\frac{GM}{r}\left(P_0(\cos\theta) - J_2\frac{R^2}{r^2}P_2(\cos\theta) + \quad\right) \tag{19}$$

where R is the radius of the Sun and J_2 is a measure of the oblateness of the Sun. Let us keep only the monopole and quadrupole terms; then

$$\kappa^2 = \frac{GM}{r^3}\left(1 - \frac{3}{2}J_2\frac{R^2}{r^2}\right) \tag{20}$$

and

$$\Omega^2 = \frac{GM}{r^3}\left(1 + \frac{3}{2}J_2\frac{R^2}{r^2}\right) \tag{21}$$

Since $\kappa \neq \Omega$ the orbit is not closed. The time taken for one rotation is $2\pi/\Omega$ whereas the time between successive approaches to aphelion is $2\pi/\kappa$. This small difference between the periods of rotation and radial oscillation shows itself as a (prograde) precession of the perihelion (or aphelion) of the planet. The magnitude of the perihelion shift per unit time is

$$\Omega - \kappa = \frac{3}{2} J_2 \frac{R^2}{r^2}. \tag{22}$$

Because of the $1/r^2$ factor, the planet whose perihelion would shift the most is Mercury. (Of course the orbit of the planet would also have to be elliptical enough for the perihelion to be distinct; fortunately, this is true for Mercury.)

The precession of the perihelion of Mercury is perhaps one of the best known observations in the history of science.

5.2. *Correction due to the General Theory of Relativity*

The precession of the perihelion of Mercury is perhaps one of the best known observations in the history of science. The total precession is 5600″ per century, of which all but 43″ had been satisfactorily explained by the middle of the 19th century (as resulting from perturbations caused by the other planets). Then, in 1916, Einstein published his General Theory of Relativity. In this theory gravity is not a force but a consequence of the curvature of spacetime around massive bodies. This, of course, is a wholly new way to conceive of gravity, but the measurable effects of the new theory can, in certain circumstances, be regarded as being due to an extra term in the gravitational potential. In this quasi-Newtonian approach the potential of a spherically symmetric body (which in Newtonian theory is $-GM/r$) becomes

$$\Phi(r) = -\frac{GM}{r} - \frac{GMl^2}{c^2 r^3}. \tag{23}$$

Comparing (18) and (23), you can see that the correction due to the oblateness of the Sun in a purely Newtonian

According to legend, Einstein was so excited by the match between observation and his theory that he could not sleep for three days.

theory of gravity is similar in its r dependence to the correction due to general relativity in the quasi-Newtonian theory for a perfectly spherical Sun. We therefore expect it too to cause a precession of the perihelion of Mercury. You can show that as a result of this correction

$$\kappa^2 = \frac{GM}{r^3}\left(1 - \frac{3l^2}{r^2c^2}\right) = \frac{GM}{r^3}\left(1 - 3\frac{v_{orb}^2}{c^2}\right), \tag{24}$$

and

$$\Omega^2 = \frac{GM}{r^3}\left(1 + 3\frac{v_{orb}^2}{c^2}\right), \tag{25}$$

(v_{orb} is the orbital velocity of the planet) so that the rate of precession of the perihelion is

$$\Omega - \kappa = \frac{3}{2}\frac{v_{orb}^2}{c^2}. \tag{26}$$

When Einstein calculated the effect of his new theory of gravity on the orbit of Mercury, he found that it predicted a precession of the perihelion of exactly 43″ per century. This was of one of the great moments in the history of science. (According to legend, Einstein was so excited by the match between observation and his theory that he could not sleep for three days.) Many years later, R H Dicke pointed out that the oblateness of the Sun could cause a similar precession. Fortunately for the general theory of relativity, the measured oblateness of the Sun is too small to account for the observed precession.

Exercise 1. Calculate the precession of the perihelion of Mercury due to the oblateness of the Sun and due to general relativity. The mass and radius of the Sun are 2×10^{30} kg and 700,000 km, and its oblateness parameter J_2 is about 10^{-5}. The radius of the orbit of Mercury is 57.9×10^6 km.

Exercise 2. The Sun is at a distance of about 27,000 light years from the centre of the Milky Way and moves in an almost circular orbit with an orbital speed of about 220 km s^{-1}. Strangely, the orbital speed is almost independent of radius for stars and other objects in the Galaxy (except near the centre). Show that the frequency of radial oscillation is about $\sqrt{2}$ times the angular frequency of rotation. What kind of gravitational potential (and mass distribution, assuming spherical symmetry) might be responsible for an orbital speed that is independent of radius?

Suggested Reading

[1] C D Murray and S F Dermott, *Solar System Dynamics*, Cambridge University Press, 1999.

[2] Online references can be found by doing a search on *epicyclic motion* on the Internet.

Errata

Resonance, Vol. 8, No. 10, pp.76-84, October 2003

Page 80: Please note that the equation (7)

$$m = \frac{3! \times 2^3}{x! \times 2^3}$$

should read as

$$m = \frac{3! \times 2^3}{x! \times 2^y}$$

Page 82: In equation (9) the summation is from (m=1, n=1) to 10 and not 6 as printed.

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